Please provide complete and well-written solutions to the following exercises.

Due February 19, in the discussion section.

Assignment 6

Exercise 1. Prove the following statements.

(a) For every real number x, the series $\sum_{j=0}^{\infty} \frac{x^j}{j!}$ is absolutely convergent. So, $\exp(x)$ exists and is a real number for every $x \in \mathbf{R}$, the power series $\sum_{j=0}^{\infty} \frac{x^j}{j!}$ has radius of convergence $R = +\infty$, and exp is an analytic function on $(-\infty, +\infty)$.
(b) exp is differentiable on $\mathbf{R}$, and for every $x \in \mathbf{R}$, we have $\exp'(x) = \exp(x)$.
(c) exp is continuous on $\mathbf{R}$, and for all real numbers $a < b$, we have $\int_a^b \exp = \exp(b) - \exp(a)$.
(d) For every $x, y \in \mathbf{R}$, we have $\exp(x+y) = \exp(x)\exp(y)$.
(e) $\exp(0) = 1$. Also, for every $x \in \mathbf{R}$, we have $\exp(x) > 0$, and $\exp(-x) = 1/\exp(x)$.
(f) exp is strictly monotone increasing. That is, whenever x, y are real numbers with $x < y$, we have $\exp(x) < \exp(y)$.

(Hints: for part (a), use the ratio test. For parts (b) and (c), use the Theorem concerning differentiation and integration of power series. For part (d), you may need the binomial formula $(x+y)^k = \sum_{j=0}^{k} \frac{k!}{j!(k-j)!} x^j y^{k-j}$. For part (e), use part (d). For part (f), use part (d) and show that $\exp(x) > 1$ for all $x > 0$.)

Exercise 2. Prove that, for every real number x, we have

$$\exp(x) = e^x.$$

(Hint: first prove the proposition for natural numbers x. Then, prove the proposition for integers. Then, prove the proposition for rational numbers. Finally, use the density of the rationals to prove the proposition for real numbers. You should find useful the identities for exponentiation by rational numbers.)

Exercise 3. Prove the following statements.

(a) For every $x \in (0, \infty)$, we have $\log'(x) = 1/x$. So, by the Fundamental Theorem of Calculus, for any $0 < a < b$, we have $\int_a^b (1/t)dt = \log(b) - \log(a)$.
(b) For all $x, y \in (0, \infty)$, we have $\log(x) + \log(y) = \log(xy)$.
(c) For all $x \in (0, \infty)$, we have $\log(1/x) = -\log x$. In particular, $\log(1) = 0$.
(d) For any $x \in (0, \infty)$ and $y \in \mathbf{R}$, we have $\log(x^y) = y \log x$.
(e) For any $x \in (-1, 1)$, we have

$$-\log(1-x) = \sum_{j=1}^{\infty} \frac{x^j}{j}.$$

In particular, log is analytic on $(0,2)$ with the power series expansion

$$\log(x) = \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j}(x-1)^j, \quad \forall\, x \in (0,2).$$

(Hints: for part (a), use the Inverse Function Theorem or Chain Rule. For parts (b),(c) and (d), use Exercise 1 and the laws of exponentiation. For part (e), let $x \in (-1,1)$, use the geometric series formula $1/(1-x) = \sum_{j=0}^{\infty} x^j$ and integrate using the Theorem concerning integration of power series.)

Exercise 4. Describe geometrically the set of complex numbers $z \in \mathbf{C}$ satisfying each of the following constraints.

- $\{z \in \mathbf{C} \colon\, |z| = 1\}$.
- $\{z \in \mathbf{C} \colon\, |z| < 1\}$.
- $\{z \in \mathbf{C} \colon z + \overline{z} = 1\}$.
- $\{z \in \mathbf{C} \colon z + \overline{z} = |z|^2\}$.

Exercise 5. Prove the following statements.

(a) For any real number x we have $\cos(x)^2 + \sin(x)^2 = 1$. In particular, $\sin(x) \in [-1,1]$ and $\cos(x) \in [-1,1]$ for all real numbers x.
(b) For any real number x, we have $\sin'(x) = \cos(x)$, and $\cos'(x) = -\sin(x)$.
(c) For any real number x, we have $\sin(-x) = -\sin(x)$ and $\cos(-x) = \cos(x)$.
(d) For any real numbers x, y we have $\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$ and $\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$.
(e) $\sin(0) = 0$ and $\cos(0) = 1$.
(f) For every real number x, we have $e^{ix} = \cos(x) + i\sin(x)$ and $e^{-ix} = \cos(x) - i\sin(x)$.

(Hints: whenever possible, write everything in terms of exponentials.)

Exercise 6. Prove the following statements

(a) For any real x we have $\cos(x+\pi) = -\cos(x)$ and $\sin(x+\pi) = -\sin(x)$. In particular, we have $\cos(x+2\pi) = \cos(x)$ and $\sin(x+2\pi) = \sin(x)$, so that sin and cos are 2π-periodic.
(b) If x is real, then $\sin(x) = 0$ if and only if x/π is an integer.
(c) If x is real, then $\cos(x) = 0$ if and only if x/π is an integer plus $1/2$.

(Hint: for part (c), it may be helpful to compute $\sin(\pi/2)$ and $\cos(\pi/2)$, and then to related $\cos(x)$ to $\sin(x+\pi/2)$.)