Please provide complete and well-written solutions to the following exercises.

Due February 12, in the discussion section.

Assignment 5

Exercise 1. Let $x \in \mathbf{R}$. Prove that the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} \sin(1+(x/n))$ converges uniformly on any compact subset of $\mathbf{R}$.

Exercise 2. (For this exercise, you can freely use facts about trigonometry that you learned in your previous courses.) Let $x \in \mathbf{R}$ and let $f\colon \mathbf{R} \to \mathbf{R}$ be the function

$$f(x) := \sum_{j=1}^{\infty} 4^{-j} \cos(32^j \pi x).$$

Note that this series is uniformly convergent by the Weierstrass M-test. So, f is a continuous function. However, at every point $x \in \mathbf{R}$, f is not differentiable, as we now discuss.

- Show that, for all positive integers j, m, we have
$$|f((j+1)/32^m) - f(j/32^m)| \geq 4^{-m}.$$
(Hint: for certain sequences of numbers $(a_j)_{j=1}^{\infty}$, use the identity
$$\sum_{j=1}^{\infty} a_j = (\sum_{j=1}^{m-1} a_j) + a_m + \sum_{j=m+1}^{\infty} a_j.$$
Also, use the fact that the cosine function is periodic with period 2π, and the summation $\sum_{j=0}^{\infty} r^j = 1/(1-r)$ for all $-1 < r < 1$. Finally, you should require the inequality: for all real numbers x, y, we have $|\cos(x) - \cos(y)| \leq |x - y|$. This inequality follows from the Mean Value Theorem or the Fundamental Theorem of Calculus.)
- Using the previous result, show that, for every $x \in \mathbf{R}$, f is not differentiable at x. (Hint: for every $x \in \mathbf{R}$ and for every positive integer m, there exists an integer j such that $j \leq 32^m x \leq j+1$.)
- Explain briefly why this result does not contradict the Corollary which concerns the differentiability of an infinite sum of differentiable functions.

Exercise 3. Let $\sum_{j=0}^{\infty} a_j (x-a)^j$ be a formal power series, and let R be its radius of convergence. Prove the following statements.

(a) (Divergence outside of the radius of convergence) If $x \in \mathbf{R}$ satisfies $|x - a| > R$, then the series $\sum_{j=0}^{\infty} a_j (x-a)^j$ is divergent at x.
(b) (Convergence inside the radius of convergence) If $x \in \mathbf{R}$ satisfies $|x - a| < R$, then the series $\sum_{j=0}^{\infty} a_j (x-a)^j$ is convergent at x.

- For the following items (c),(d) and (e), we assume that $R > 0$. Then, let $f\colon (a-R, a+R)$ be the function $f(x) = \sum_{j=0}^{\infty} a_j (x-a)^j$, which exists by part (b).

(c) (Uniform convergence on compact intervals) For any $0 < r < R$, we know that the series $\sum_{j=0}^{\infty} a_j (x-a)^j$ converges uniformly to f on $[a-r, a+r]$. In particular, f is continuous on $(a-R, a+R)$ (since uniform convergence preserves continuity).

(d) (Differentiation of power series) The function f is differentiable on $(a-R, a+R)$. For any $0 < r < R$, the series $\sum_{j=0}^{\infty} j a_j (x-a)^{j-1}$ converges uniformly to f' on the interval $[a-r, a+r]$.

(e) (Integration of power series) For any closed interval $[y, z]$ contained in $(a-R, a+R)$, we have

$$\int_y^z f = \sum_{j=0}^{\infty} a_j \frac{(z-a)^{n+1} - (y-a)^{n+1}}{n+1}.$$

(Hints: for parts (a),(b), use the root test. For part (c), use the Weierstrass M-test. For part (d), use the Theorem on uniform convergence and differentiability. For part (e), use the Theorem concerning integrability of the infinite sum of certain integrable functions.)

Exercise 4. Give examples of formal power series centered at 0 with radius of convergence $R = 1$ such that

- The series diverges at $x = 1$ and at $x = -1$.
- The series diverges at $x = 1$ and converges at $x = -1$.
- The series converges at $x = 1$ and diverges at $x = -1$.
- The series converges at $x = 1$ and at $x = -1$.

Exercise 5. Let $a \in \mathbf{R}$ and let $r > 0$. Let f be a function that is real analytic on $(a-r, a+r)$, with the power series expansion

$$f(x) = \sum_{j=0}^{\infty} a_j (x-a)^j, \quad \forall\, x \in (a-r, a+r).$$

Show that, for any integer $k \geq 0$, the function f is k times differentiable on $(a-r, a+r)$, and the k^{th} derivative is given by

$$f^{(k)}(x) = \sum_{j=0}^{\infty} a_{j+k} (j+1)(j+2)\cdots(j+k)(x-a)^j, \quad \forall\, x \in (a-r, a+r).$$

Exercise 6 (**Taylor's formula**)**.** Let $a \in \mathbf{R}$ and let $r > 0$. Let f be a function that is real analytic on $(a-r, a+r)$, with the power series expansion

$$f(x) = \sum_{j=0}^{\infty} a_j (x-a)^j, \quad \forall\, x \in (a-r, a+r).$$

Show that, for any integer $k \geq 0$, we have

$$f^{(k)}(a) = k! a_k,$$

where $k! = 1 \times 2 \times \cdots \times k$, and we denote $0! := 1$. In particular, we have **Taylor's formula**

$$f(x) = \sum_{j=0}^{\infty} \frac{f^{(j)}(a)}{j!} (x-a)^j, \quad \forall\, x \in (a-r, a+r).$$

Exercise 7. Define a function $f\colon \mathbf{R} \to \mathbf{R}$ by $f(0) := 0$ and $f(x) := e^{-1/x^2}$ for $x \neq 0$. Show that f is infinitely differentiable, but $f^{(k)}(0) = 0$ for all $k \geq 0$. So, being infinitely differentiable does not imply that f is equal to its Taylor series. (You may freely use properties of the exponential function that you have learned before.)