Please provide complete and well-written solutions to the following exercises.

Due February 5, in the discussion section.

Assignment 4

Exercise 1. Let (X, d_X) and (Y, d_Y) be metric spaces. Let $(f_j)_{j=1}^{\infty}$ be a sequence of functions from X to Y. Let $f\colon X \to Y$ be another function. Suppose $(f_j)_{j=1}^{\infty}$ converges uniformly to f on X. Suppose also that, for each $j \geq 1$, we know that f_j is bounded. Show that f is also bounded.

Exercise 2. Let (X, d_X) and (Y, d_Y) be metric spaces. Let $B(X; Y)$ denote the set of functions $f\colon X \to Y$ that are bounded. Let $f, g \in B(X; Y)$. We define the metric $d_\infty\colon B(X;Y) \times B(X;Y) \to [0, \infty)$ by

$$d_\infty(f, g) := \sup_{x \in X} d_Y(f(x), g(x)).$$

Show that the space $(B(X;Y), d_\infty)$ is a metric space.

Exercise 3. Let (X, d_X) and (Y, d_Y) be metric spaces. Let $(f_j)_{j=1}^{\infty}$ be a sequence of functions in $B(X;Y)$. Let $f \in B(X;Y)$. Show that $(f_j)_{j=1}^{\infty}$ converges uniformly to f on X if and only if $(f_j)_{j=1}^{\infty}$ converges to f in the metric $d_{B(X;Y)}$.

Exercise 4. Let (X, d_X) be a metric space, and let (Y, d_Y) be a complete metric space. Then the space $(C(X;Y), d_{B(X;Y)}|_{C(X;Y)\times C(X;Y)})$ is a complete subspace of $B(X;Y)$. That is, every Cauchy sequence of functions in $C(X;Y)$ converges to a function in $C(X;Y)$.

Exercise 5. Let $x \in (-1, 1)$. For each integer $j \geq 1$, define $f_j(x) := x^j$. Show that the series $\sum_{j=1}^{\infty} f_j$ converges pointwise, but not uniformly, on $(-1, 1)$ to the function $f(x) = x/(1-x)$. Also, for any $0 < t < 1$, show that the series $\sum_{j=1}^{\infty} f_j$ converges uniformly to f on $[-t, t]$.

Exercise 6. Let X be a set. Show that $\|\cdot\|_\infty$ is a norm on the space $B(X; \mathbf{R})$.

Exercise 7 (Weierstrass M-test). Let (X, d) be a metric space and let $(f_j)_{j=1}^{\infty}$ be a sequence of bounded real-valued continuous functions on X such that the series (of real numbers) $\sum_{j=1}^{\infty} \|f_j\|_\infty$ is absolutely convergent. Show that the series $\sum_{j=1}^{\infty} f_j$ converges uniformly to some continuous function $f\colon X \to \mathbf{R}$. (Hint: first, show that the partial sums $\sum_{j=1}^{J} f_j$ form a Cauchy sequence in $C(X; \mathbf{R})$. Then, use Exercise 4 and the completeness of the real line $\mathbf{R}$.)

Exercise 8. Let $a < b$ be real numbers. For each integer $j \geq 1$, let $f_j\colon [a, b] \to \mathbf{R}$ be a Riemann integrable function on $[a, b]$. Suppose $\sum_{j=1}^{\infty} f_j$ converges uniformly on $[a, b]$. Then $\sum_{j=1}^{\infty} f_j$ is also Riemann integrable, and

$$\sum_{j=1}^{\infty} \int_a^b f_j = \int_a^b \sum_{j=1}^{\infty} f_j.$$

Exercise 9. Let $a < b$. For every integer $j \geq 1$, let $f_j \colon [a,b] \to \mathbf{R}$ be a differentiable function whose derivative $(f_j)' \colon [a,b] \to \mathbf{R}$ is continuous. Assume that the derivatives $(f_j)'$ converge uniformly to a function $g \colon [a,b] \to \mathbf{R}$ as $j \to \infty$. Assume also that there exists a point $x_0 \in [a,b]$ such that $\lim_{j\to\infty} f_j(x_0)$ exists. Then the functions f_j converge uniformly to a differentiable function f as $j \to \infty$, and $f' = g$.

Exercise 10. Let $a < b$. For every integer $j \geq 1$, let $f_j \colon [a,b] \to \mathbf{R}$ be a differentiable function whose derivative $f_j' \colon [a,b] \to \mathbf{R}$ is continuous. Assume that the series of real numbers $\sum_{j=1}^{\infty} \|f_j'\|_{\infty}$ is absolutely convergent. Assume also that there exists $x_0 \in [a,b]$ such that the series of real numbers $\sum_{j=1}^{\infty} f_j(x_0)$ converges. Then the series $\sum_{j=1}^{\infty} f_j$ converges uniformly on $[a,b]$ to a differentiable function. Moreover, for all $x \in [a,b]$,

$$\frac{d}{dx}\sum_{j=1}^{\infty} f_j(x) = \sum_{j=1}^{\infty} \frac{d}{dx} f_j(x)$$