Please provide complete and well-written solutions to the following exercises.

Due January 29, in the discussion section.

# Assignment 3

**Exercise 1.** Determine which of the following subsets of $\mathbf{R}^2$ are compact. Justify your answers. (As usual, if we do not specify a metric on $\mathbf{R}^2$, we mean $\mathbf{R}^2$ with the standard Euclidean metric $d_{\ell_2}$.)

- $\{(x,y)\in\mathbf{R}^2\colon x^2+y^2=3\}$.
- $\{(x,y)\in\mathbf{R}^2\colon 0\leq xy\leq 1\}$.
- $\{(1,1/n)\in\mathbf{R}^2\colon n\in\mathbf{N}\}$.
- $\{(x,y)\in\mathbf{R}^2\colon x^2+y^2<3\}$.
- $\{(x,y)\in\mathbf{R}^2\colon 0\leq x\leq 1,\, 0\leq y\leq x^2\}$.

**Exercise 2.** Let $(X,d_X)$ and $(Y,d_Y)$ be vector spaces. Let $f\colon X\to Y$ be a continuous function. Let $E$ be a connected subset of $X$. Show that $f(E)$ is connected.

**Exercise 3.** Using the previous exercise, prove the Intermediate Value Theorem: Let $(X,d)$ be a metric space. Let $f\colon X\to\mathbf{R}$ be a continuous function. Let $E$ be a connected subset of $X$ and let $a,b$ be any two elements of $E$. Let $y$ be a real number between $f(a)$ and $f(b)$, so that either $f(a)\leq y\leq f(b)$ or $f(b)\leq y\leq f(a)$. Then there exists $c\in E$ such that $f(c)=y$.

**Exercise 4.** Let $(X,d_X)$ and $(Y,d_Y)$ be metric spaces, let $E$ be a subset of $X$, let $f\colon X\to Y$ be a function, let $x_0\in X$ be an adherent point of $E$, and let $L\in Y$. Show that the following statements are equivalent.

- $\lim_{x\to x_0;x\in E}f(x)=L$.
- For any sequence $(x^{(j)})_{j=1}^{\infty}$ in $E$ which converges to $x_0$ with respect to the metric $d_X$, the sequence $(f(x^{(j)}))_{j=1}^{\infty}$ converges to $L$ with respect to the metric $d_Y$.

**Exercise 5.** Let $(X,d_X)$ and $(Y,d_Y)$ be metric spaces. Let $(f_j)_{j=1}^{\infty}$ be a sequence of functions from $X$ to $Y$. Let $f\colon X\to Y$ be another function. Let $x_0\in X$. Suppose $f_j$ converges uniformly to $f$ on $X$. Suppose that, for each $j\geq 1$, we know that $f_j$ is continuous at $x_0$. Show that $f$ is also continuous at $x_0$. Hint: it is probably easiest to use the $\varepsilon-\delta$ definition of continuity. Once you do this, you may require the triangle inequality in the form

$$d_Y(f(x),f(x_0))\leq d_Y(f(x),f_j(x))+d_Y(f_j(x),f_j(x_0))+d_Y(f_j(x_0),f(x_0)).$$