Please provide complete and well-written solutions to the following exercises.

Due January 22, in the discussion section.

Assignment 2

Exercise 1. Let (X, d) be a compact metric space. Show that (X, d) is both complete and bounded. (Hint: prove each property separately, and use argument by contradiction.)

Exercise 2. Let n be a positive integer. Let $(\mathbf{R}^n, d)$ denote Euclidean space with the metric $d = d_{\ell_2}$ or $d = d_{\ell_1}$. Let E be a subset of $\mathbf{R}^n$. Show that E is compact if and only if E is both closed and bounded. (Hint: use Bolzano-Weierstrass in $\mathbf{R}^n$.)

Exercise 3. Let (X, d) be a metric space, and let $K_1, K_2, \ldots$ be a sequence of nonempty compact subsets of X such that

$$K_1 \supseteq K_2 \supseteq K_3 \supseteq \cdots .$$

Show that the intersection $\cap_{j=1}^{\infty} K_j$ is nonempty. (Hint: first, work in the compact metric space $(K_1, d|_{K_1 \times K_1})$. Then, consider the sets $K_1 \setminus K_j$ which are open in K_1. Assume for the sake of contradiction that $\cap_{j=1}^{\infty} K_j = \emptyset$. Then apply the Open Cover Characterization of compactness.)

Exercise 4. Let (X, d_X) and let (Y, d_Y) be metric spaces. Let $f\colon X \to Y$ be a function. Show that the following two statements are equivalent.

- f is continuous at x_0.
- If we have a sequence $(x^{(j)})_{j=1}^{\infty}$ in X which converges to x_0 with respect to d_X, then the sequence $(f(x^{(j)}))_{j=1}^{\infty}$ converges to $f(x_0)$ with respect to the metric d_Y.

Exercise 5. Let (X, d_X) and let (Y, d_Y) be metric spaces. Let $f\colon X \to Y$ be a function. Show that the following four statements are equivalent.

- f is continuous at x_0, for all $x_0 \in X$.
- For all $x_0 \in X$, if we have a sequence $(x^{(j)})_{j=1}^{\infty}$ in X which converges to x_0 with respect to d_X, then the sequence $(f(x^{(j)}))_{j=1}^{\infty}$ converges to $f(x_0)$ with respect to the metric d_Y.
- For all open sets W in Y, the set $f^{-1}(W) = \{x \in X : f(x) \in W\}$ is an open set in X.
- For all closed sets V in Y, the set $f^{-1}(V)$ is a closed set in X.

Exercise 6. Let (X, d_X), (Y, d_Y) and (Z, d_Z) be metric spaces. Let $f\colon (X, d_X) \to (Y, d_Y)$ be a continuous function and let $g\colon (Y, d_Y) \to (Z, d_Z)$ be a continuous function. Show that $g \circ f\colon (X, d_X) \to (Z, d_Z)$ is a continuous function.

Exercise 7. Give an example of a continuous function $f\colon \mathbf{R} \to \mathbf{R}$ and of an open set W such that $f(W)$ is not open.

Exercise 8. Give an example of a continuous function $f\colon \mathbf{R} \to \mathbf{R}$ and of a closed set W such that $f(W)$ is not closed.

Exercise 9. Let (X, d_X) and (Y, d_Y) be metric spaces. Let $f\colon (X, d_X) \to (Y, d_Y)$ be a continuous function. Suppose $K \subseteq X$ is a compact set. Show that $f(K) = \{f(x)\colon x \in K\}$ is also a compact set.

Exercise 10. Using the previous exercise, prove the Maximum Principle: Let K be a closed and bounded subset of $\mathbf{R}^n$, and let $f\colon K \to \mathbf{R}$ be a continuous function. Then there exist points $a, b \in K$ such that f attains its maximum at a and f attains its minimum at b. (Hint: consider the numbers $\sup_{x\in K} f(x)$ and $\inf_{x\in K} f(x)$.)

Exercise 11. Let n be a positive integer. Let $\|\cdot\|$ and let $\|\cdot\|'$ be two norms on $\mathbf{R}^n$. Prove that these norms are equivalent. That is, there exist constants $C, c > 0$ such that, for all $x \in \mathbf{R}^n$, we have $c\,\|x\|' \leq \|x\| \leq C\,\|x\|'$. Consequently, any two norms on $\mathbf{R}^n$ are equivalent. (Hint: there are a few ways to solve this problem, but it is difficult to avoid circular reasoning. Here is one way to solve the problem.

- Note that it suffices to assume that $\|x\|' = \|x\|_{\ell_\infty}$.
- Let $(e_1, \ldots, e_n)$ denote the standard basis of $\mathbf{R}^n$, and prove that

$$\|x\| \leq (\sum_{i=1}^{n} \|e_i\|)\,\|x\|_{\ell_\infty}\,.$$

- Consider $f\colon \mathbf{R}^n \to \mathbf{R}$ defined by $f(x) := \|x\|$. From the previous item, f is a continuous function from $(\mathbf{R}^n, d_{\ell_\infty})$ into $\mathbf{R}$. Let S denote the unit cube $S := \{x \in \mathbf{R}^n\colon \|x\|_{\ell_\infty} = 1\}$. Using that S is compact with respect to d_{ℓ_∞}, now apply the maximum principle to f on the set S.

Remark 1. There exist infinite dimensional vector spaces with norms that are not equivalent.