Please provide complete and well-written solutions to the following exercises.

Due December 11, in the discussion section.

Assignment 9

Exercise 1. Let $a < b$ be real numbers, and let $f, g\colon [a,b] \to \mathbf{R}$ be Riemann integrable functions on $[a,b]$. Then

(i) The function $f+g$ is Riemann integrable on $[a,b]$, and $\int_a^b (f+g) = (\int_a^b f) + (\int_a^b g)$.
(ii) For any real number c, cf is Riemann integrable on $[a,b]$, and $\int_a^b (cf) = c(\int_a^b f)$.
(iii) The function $f-g$ is Riemann integrable on $[a,b]$, and $\int_a^b (f-g) = (\int_a^b f) - (\int_a^b g)$.
(iv) If $f(x) \geq 0$ for all $x \in [a,b]$, then $\int_a^b f \geq 0$.
(v) If $f(x) \geq g(x)$ for all $x \in [a,b]$, then $\int_a^b f \geq \int_a^b g$.
(vi) If there exists a real number c such that $f(x) = c$ for $x \in [a,b]$, then $\int_a^b f = c(b-a)$.
(vii) Let c, d be real numbers such that $c \leq a < b \leq d$. Then $[c,d]$ contains $[a,b]$. Define $F(x) := f(x)$ for all $x \in [a,b]$ and $F(x) := 0$ otherwise. Then F is Riemann integrable on $[c,d]$, and $\int_c^d F = \int_a^b f$.
(viii) Let c be a real number such that $a < c < b$. Then $f|_{[a,c]}$ and $f|_{[c,b]}$ are Riemann integrable on $[a,c]$ and $[c,b]$ respectively, and

$$\int_a^b f = \int_a^c f|_{[a,c]} + \int_c^b f|_{[c,b]}.$$

Exercise 2. Let $a < b$ be real numbers. Let $f\colon [a,b] \to \mathbf{R}$ be a bounded function. Let $c \in [a,b]$. Assume that, for each $\delta > 0$, we know that f is Riemann integrable on the set $\{x \in [a,b]\colon\ |x-c| \geq \delta\}$. Then f is Riemann integrable on $[a,b]$.

Exercise 3. Find a function $f\colon [0,1] \to \mathbf{R}$ such that f is not Riemann integrable on $[0,1]$, but such that $|f|$ is Riemann integrable on $[0,1]$.

Exercise 4. Let $a < b$ be real numbers. Let $f\colon [a,b] \to \mathbf{R}$ be a bounded function. So, there exists a real number M such that $|f(x)| \leq M$ for all $x \in [a,b]$. Let P be a partition of $[a,b]$.

- Using the identity $\alpha^2 - \beta^2 = (\alpha+\beta)(\alpha-\beta)$, where $\alpha, \beta \in \mathbf{R}$, show that
$$U(f^2, P) - L(f^2, P) \leq 2M(U(f,P) - L(f,P)).$$
- Show that if f is Riemann integrable on $[a,b]$, then f^2 is also Riemann integrable on $[a,b]$.
- Let $f, g\colon [a,b] \to \mathbf{R}$ be Riemann integrable functions on $[a,b]$. Using the identity $4\alpha\beta = (\alpha+\beta)^2 - (\alpha-\beta)^2$, where $\alpha, \beta \in \mathbf{R}$, show that fg is Riemann integrable on $[a,b]$.

Exercise 5. Let $f\colon [0,1] \to [0,\infty)$ be a continuous function such that $\int_0^1 f = 0$. Prove that $f(x) = 0$ for all $x \in [0,1]$.

Exercise 6. The following exercise deals with metric properties of the space of Riemann integrable functions.

- Let α, β be real numbers. Prove that $\alpha\beta \leq (\alpha^2 + \beta^2)/2$. Now, let $a < b$ be real numbers, and let $f, g\colon [a,b] \to \mathbf{R}$ be two Riemann integrable functions. Assume that $\int_a^b f^2 = 1$ and $\int_a^b g^2 = 1$. (Recall that since f, g are Riemann integrable, we know that f^2, g^2 and fg are also Riemann integrable by Exercise 4.) Prove that
$$\int_a^b fg \leq 1.$$
- Let $a < b$ be real numbers, and let $f, g\colon [a,b] \to \mathbf{R}$ be two Riemann integrable functions. Prove the Cauchy-Schwarz inequality:
$$\left|\int_a^b fg\right| \leq \left(\int_a^b f^2\right)^{1/2} \left(\int_a^b g^2\right)^{1/2}$$
- Let $a < b$ be real numbers, and let $f, g, h\colon [a,b] \to \mathbf{R}$ be Riemann integrable functions. Define
$$d(f,g) := \left(\int_a^b (f-g)^2\right)^{1/2}.$$
Prove the triangle inequality for d. That is, show that
$$d(f,g) \leq d(f,h) + d(h,g).$$