Please provide complete and well-written solutions to the following exercises.

Due November 20, in the discussion section.

Assignment 7

Exercise 1. Let $(a_n)_{n=m}^{\infty}$ be a sequence of real numbers converging to 0. Show that $(|a_n|)_{n=m}^{\infty}$ also converges to zero.

Exercise 2. Let $a < b$ be real numbers. Let I be any of the four intervals (a, b), $(a, b]$, $[a, b)$ or $[a, b]$. Then the closure of I is $[a, b]$.

Exercise 3. Let X be a subset of $\mathbf{R}$, let $f : X \to \mathbf{R}$ be a function, let E be a subset of X, let x_0 be an adherent point of E, and let L be a real number. Then the following two statements are equivalent. (That is, one statement is true if and only if the other statement is true.)

- f converges to L at x_0 in E.
- For every sequence $(a_n)_{n=0}^{\infty}$ which consists entirely of elements of E, and which converges to x_0, the sequence $(f(a_n))_{n=0}^{\infty}$ converges to L.

Exercise 4. Let X be a subset of $\mathbf{R}$, let $f : X \to \mathbf{R}$ be a function, let E be a subset of X, let x_0 be an adherent point of E, let L be a real number, and let δ be a positive real number. Then the following two statements are equivalent:

- $\lim_{x \to x_0; x \in E} f(x) = L$.
- $\lim_{x \to x_0; x \in E \cap (x_0 - \delta, x_0 + \delta)} f(x) = L$.

Exercise 5. Let X be a subset of $\mathbf{R}$, let $f : X \to \mathbf{R}$ be a function, and let $x_0 \in X$. Then the following three statements are equivalent.

- f is continuous at x_0
- For every sequence $(a_n)_{n=0}^{\infty}$ consisting of elements of X such that $\lim_{n \to \infty} a_n = x_0$, we have $\lim_{n \to \infty} f(a_n) = f(x_0)$.
- For every $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that, for all $x \in X$ with $|x - x_0| < \delta$, we have $|f(x) - f(x_0)| < \varepsilon$.

Exercise 6. Let X, Y be subsets of $\mathbf{R}$. Let $f : X \to Y$ and let $g : Y \to \mathbf{R}$ be functions. Let $x_0 \in X$. If f is continuous at x_0, and if g is continuous at $f(x_0)$, then $g \circ f$ is continuous at x_0.

Exercise 7. Let $a < b$ be real numbers. Let $f : [a, b] \to \mathbf{R}$ be a continuous function on $[a, b]$. Let $M := \sup_{x \in [a,b]} f(x)$ be the maximum value of f on $[a, b]$, and let $m := \inf_{x \in [a,b]} f(x)$ be the minimum value of f on $[a, b]$. Let y be a real number such that $m \leq y \leq M$. Then there exists $c \in [a, b]$ such that $f(c) = y$. Moreover, $f([a, b]) = [m, M]$.

Exercise 8. Let $a < b$ be real numbers, and let $f\colon [a,b] \to \mathbf{R}$ be a function which is both continuous and strictly monotone increasing. Then f is a bijection from $[a,b]$ to $[f(a), f(b)]$, and the inverse function $f^{-1}\colon [f(a), f(b)] \to [a,b]$ is also continuous and strictly monotone increasing. (Hint: To prove that f^{-1} is continuous, use the ε–δ definition of continuity.)

Exercise 9. Let $(a_n)_{n=m}^{\infty}, (b_n)_{n=m}^{\infty}$ be two sequences of real numbers. Then $(a_n)_{n=m}^{\infty}$ and $(b_n)_{n=m}^{\infty}$ are equivalent if and only if $\lim_{n\to\infty}(a_n - b_n) = 0$.

Exercise 10. Let $a < b$ be real numbers, and let $f\colon [a,b] \to \mathbf{R}$ be a function. Assume that there exists a real number $L > 0$ such that, for all $x, y \in [a,b]$, we have $|f(x) - f(y)| \leq L\,|x - y|$. Such an f is called **Lipschitz continuous**. Prove that f is continuous. Then, find a continuous function that is not Lipschitz continuous.