Please provide complete and well-written solutions to the following exercises.

Due November 13, in the discussion section.

Assignment 6

Exercise 1. Let $\sum_{n=m}^{\infty} a_n$ be a formal series of real numbers. Then $\sum_{n=m}^{\infty} a_n$ converges if and only if: for every real number $\varepsilon > 0$, there exists an integer $N \geq M$ such that, for all $p, q \geq N$,

$$\left|\sum_{n=p}^{q} a_n\right| < \varepsilon.$$

(Hint: recall that a sequence is convergent if and only if it is a Cauchy sequence.)

Exercise 2. Let $\sum_{n=m}^{\infty} a_n$ be a formal series of real numbers. If $\sum_{n=m}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$. Note that the contrapositive says: if a_n does not converge to zero as $n \to \infty$, then $\sum_{n=m}^{\infty} a_n$ does not converge. (Hint: use Exercise 1.)

Exercise 3. Let $\sum_{n=m}^{\infty} a_n$ be a formal series of real numbers. If this series is absolutely convergent, then it is convergent. Moreover,

$$\left|\sum_{n=m}^{\infty} a_n\right| \leq \sum_{n=m}^{\infty} |a_n| .$$

Exercise 4.

- Let $\sum_{n=m}^{\infty} a_n$ be a series of real numbers converging to x, and let $\sum_{n=m}^{\infty} b_n$ be a series of real numbers converging to y. Then $\sum_{n=m}^{\infty}(a_n + b_n)$ is a convergent series that converges to $x + y$. That is,

$$\sum_{n=m}^{\infty}(a_n + b_n) = (\sum_{n=m}^{\infty} a_n) + (\sum_{n=m}^{\infty} b_n).$$

- Let $\sum_{n=m}^{\infty} a_n$ be a series of real numbers converging to x, and let c be a real number. Then $\sum_{n=m}^{\infty}(ca_n)$ is a convergent series that converges to cx. That is,

$$\sum_{n=m}^{\infty}(ca_n) = c(\sum_{n=m}^{\infty} a_n).$$

- Let $\sum_{n=m}^{\infty} a_n$ be a series of real numbers, and let k be a natural number. If one of the two series $\sum_{n=m}^{\infty} a_n$ or $\sum_{n=m+k}^{\infty} a_n$ converges, then the other also converges, and we have

$$\sum_{n=m}^{\infty} a_n = (\sum_{n=m}^{m+k-1} a_n) + (\sum_{n=m+k}^{\infty} a_n).$$

- Let $\sum_{n=m}^{\infty} a_n$ be a series of real numbers converging to x, and let k be an integer. Then $\sum_{n=m+k}^{\infty} a_{n-k}$ also converges to x.

Exercise 5. Let $\sum_{n=m}^{\infty} a_n, \sum_{n=m}^{\infty} b_n$ be formal series of real numbers. Assume that $|a_n| \leq b_n$ for all $n \geq m$. If $\sum_{n=m}^{\infty} b_n$ is convergent, then $\sum_{n=m}^{\infty} a_n$ is absolutely convergent. Moreover,

$$\left| \sum_{n=m}^{\infty} a_n \right| \leq \sum_{n=m}^{\infty} |a_n| \leq \sum_{n=m}^{\infty} b_n.$$

Exercise 6. For any $n \in \mathbf{N}$, define $a_n := (-1)^{n+1}/(n+1)$. Find a bijection $g\colon \mathbf{N} \to \mathbf{N}$ such that the series $\sum_{n=0}^{\infty} a_{g(n)}$ diverges.

Exercise 7. Let $(b_n)_{n=m}^{\infty}$ be a sequence of positive numbers. Then

$$\liminf_{n\to\infty} \frac{b_{n+1}}{b_n} \leq \liminf_{n\to\infty} b_n^{1/n}.$$

Exercise 8. Let $(a_n)_{n=0}^{\infty}, (b_n)_{n=0}^{\infty}, (c_n)_{n=0}^{\infty}$ be sequences of real numbers. Then $(a_n)_{n=0}^{\infty}$ is a subsequence of $(a_n)_{n=0}^{\infty}$. Also, if $(b_n)_{n=0}^{\infty}$ is a subsequence of $(a_n)_{n=0}^{\infty}$, and if $(c_n)_{n=0}^{\infty}$ is a subsequence of $(b_n)_{n=0}^{\infty}$, then $(c_n)_{n=0}^{\infty}$ is a subsequence of $(a_n)_{n=0}^{\infty}$.

Exercise 9. Give an example of two convergent series of real numbers $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ such that the series $\sum_{n=0}^{\infty}(a_n b_n)$ is not convergent.

Exercise 10. Let $(a_n)_{n=0}^{\infty}$ be a sequence of real numbers, and let L be a real number.

- If the sequence $(a_n)_{n=0}^{\infty}$ converges to L, then every subsequence of $(a_n)_{n=0}^{\infty}$ converges to L.
- Conversely, if every subsequence of $(a_n)_{n=0}^{\infty}$ converges to L, then $(a_n)_{n=0}^{\infty}$ itself converges to L.