Please provide complete and well-written solutions to the following exercises.

Due November 6, in the discussion section.

Assignment 5

Exercise 1. Let $(a_n)_{n=m}^{\infty}$, $(b_n)_{n=m}^{\infty}$ be sequences of real numbers such that $\limsup_{n\to\infty} a_n$ and $\limsup_{n\to\infty} b_n$ are finite. Prove:

$$\limsup_{n\to\infty}(a_n+b_n) \leq (\limsup_{n\to\infty} a_n) + (\limsup_{n\to\infty} b_n).$$

Exercise 2. Let $(a_n)_{n=m}^{\infty}$, $(b_n)_{n=m}^{\infty}$ be sequences of real numbers. Assume that $a_n \leq b_n$ for all $n \geq m$. Prove:

- $\sup(a_n)_{n=m}^{\infty} \leq \sup(b_n)_{n=m}^{\infty}$.
- $\inf(a_n)_{n=m}^{\infty} \leq \inf(b_n)_{n=m}^{\infty}$.
- $\limsup_{n\to\infty} a_n \leq \limsup_{n\to\infty} b_n$.
- $\liminf_{n\to\infty} a_n \leq \liminf_{n\to\infty} b_n$.

Exercise 3. Let $(a_n)_{n=m}^{\infty}$, $(b_n)_{n=m}^{\infty}$, $(c_n)_{n=m}^{\infty}$ be sequences of real numbers such that there exists a natural number M such that, for all $n \geq M$,

$$a_n \leq b_n \leq c_n.$$

Assume that $(a_n)_{n=m}^{\infty}$ and $(c_n)_{n=m}^{\infty}$ converge to the same limit L. Prove that $(b_n)_{n=m}^{\infty}$ converges to L. (Hint: use the previous exercise.)

Exercise 4. Let $x, y > 0$ be positive real numbers, and let $n, m \geq 1$ be positive integers. Prove:

(i) If $y = x^{1/n}$, then $y^n = x$.
(ii) If $y^n = x$, then $y = x^{1/n}$.
(iii) $x^{1/n}$ is a positive real number.
(iv) $x > y$ if and only if $x^{1/n} > y^{1/n}$.
(v) If $x > 1$ then $x^{1/n}$ decreases when n increases. If $x < 1$, then $x^{1/n}$ increases when n increases. If $x = 1$, then $x^{1/n} = 1$ for all positive integers n.
(vi) $(xy)^{1/n} = x^{1/n}y^{1/n}$.
(vii) $(x^{1/n})^{1/m} = x^{1/(nm)}$.

Exercise 5. Let $x, y > 0$ be positive real numbers, and let q, r be rational numbers. Prove:

(i) x^q is a positive real number.
(ii) $x^{q+r} = x^q x^r$ and $(x^q)^r = x^{qr}$.
(iii) $x^{-q} = 1/x^q$.
(iv) If $q > 0$, then $x > y$ if and only if $x^q > y^q$.
(v) If $x > 1$, then $x^q > x^r$ if and only if $q > r$. If $x < 1$, then $x^q > x^r$ if and only if $q < r$.

Exercise 6. Let $-1 < x < 1$. Show that $\lim_{n\to\infty} x^n = 0$. Using the identity $(1/x^n)x^n = 1$ for $x > 1$, conclude that x^n does not converge as $n \to \infty$ for $x > 1$.

Exercise 7. For any $x > 0$, show that $\lim_{n\to\infty} x^{1/n} = 1$. (Hint: first, given any $\varepsilon > 0$, show that $(1+\varepsilon)^n$ has no real upper bound M, as $n \to \infty$. To prove this claim, set $x = 1/(1+\varepsilon)$ and use Exercise 6. Now, with this preliminary claim, show that for any $\varepsilon > 0$ and for any real M, there exists a positive integer n such that $M^{1/n} < 1+\varepsilon$. Now, use these two claims, and consider the cases $x > 1$ and $x < 1$ separately.)

Exercise 8. Let $m \leq n < p$ be integers, let $(a_i)_{i=m}^n, (b_i)_{i=m}^n$ be a sequences of real numbers, let k be an integer, and let c be a real number. Prove:

- $$\sum_{i=m}^n a_i + \sum_{i=n+1}^p a_i = \sum_{i=m}^p a_i.$$
- $$\sum_{i=m}^n a_i = \sum_{j=m+k}^{n+k} a_{j-k}.$$
- $$\sum_{i=m}^n (a_i + b_i) = (\sum_{i=m}^n a_i) + (\sum_{i=m}^n b_i).$$
- $$\sum_{i=m}^n (ca_i) = c(\sum_{i=m}^n a_i).$$
- $$\left|\sum_{i=m}^n a_i\right| \leq \sum_{i=m}^n |a_i|.$$
- $$\text{If } a_i \leq b_i \text{ for all } m \leq i \leq n, \text{ then } \sum_{i=m}^n a_i \leq \sum_{i=m}^n b_i.$$

Exercise 9. Let X be a finite set of cardinality $n \in \mathbf{N}$. Let $f\colon X \to \mathbf{R}$ be a function. Then for any two bijections $g, h\colon \{1, 2, \ldots, n\} \to X$, show that $\sum_{i=1}^n f(g(i)) = \sum_{i=1}^n f(h(i))$.