Please provide complete and well-written solutions to the following exercises.

Due October 9, in the discussion section.

Assignment 1

Exercise 1. Read this: http://math.berkeley.edu/~hutching/teach/proofs.pdf

Exercise 2. Let A, B be subsets of some set X. Define $A^c := \{x \in X : x \notin A\}$. Prove:

$$(A \cap B)^c = A^c \cup B^c.$$

(Hint: To build some intuition, draw a picture. Now to do the full proof, let P denote the statement $x \in A$, and let Q denote the statement $x \in B$. Construct a truth table for P and Q.)

Exercise 3. Using the Peano axioms, show that the sum of two natural numbers is a natural number.

Exercise 4. Using the Peano axioms, show that addition is associative. That is, given natural numbers x, y, z, we have $x + (y + z) = (x + y) + z$. (Hint: fix two of the variables, and induct on the third.) (Note: you can use Lemma 2.9 from the notes.)

Remark 1. From this point, you can freely use basic properties of arithmetic. (That is, you no longer need to explicitly use Peano's axioms.)

Exercise 5. Let a, b, c be natural numbers. Using the definition of the order on the natural numbers, prove the following properties.

(1) $a \geq a$.
(2) If $a \geq b$ and $b \geq c$, then $a \geq c$.
(3) If $a \geq b$ and $b \geq a$, then $a = b$.
(4) $a \geq b$ if and only if $a + c \geq b + c$.
(5) $a < b$ if and only if $a + c < b + c$.

Exercise 6 (The Euclidean Algorithm). Let n be a natural number and let q be a positive natural number. Show that there exist natural numbers m, r such that $0 \leq r < q$ and such that $n = mq + r$. (Hint: fix q and induct on n.)

Exercise 7. Prove the principle of infinite descent. Let $p_0, p_1, p_2, \ldots$ be an infinite sequence of natural numbers such that $p_0 > p_1 > p_2 > \cdots$. Prove that no such sequence exists. (Hint: Assume by contradiction that such a sequence exists. Then prove by induction that for all natural numbers n, N, we have $p_n \geq N$. Use this fact to obtain a contradiction.)

Exercise 8. Find a set of integers a_{ij} where $i, j \in \mathbf{N}$ such that $\sum_{i=1}^{\infty}(\sum_{j=1}^{\infty} a_{ij}) = 0$, but such that $\sum_{j=1}^{\infty}(\sum_{i=1}^{\infty} a_{ij}) = 1$. (Hint: an example exists where most of the numbers are zero, and the remaining numbers are $+1$ or -1. It may also help to arrange the numbers in a matrix.)