7: FOURIER SERIES

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Contents

1. Review

Exercise 1.1. Let $(X,\langle\cdot,\cdot\rangle)$ be a (real or complex) inner product space. Define $\|\cdot\|: X\to[0,\infty)$ by $\|x\|:=\sqrt{\langle x,x\rangle}$. Then $(X,\|\cdot\|)$ is a normed linear space. Consequently, if we define $d\colon X\times X\to[0,\infty)$ by $d(x,y):=\sqrt{\langle (x-y),(x-y)\rangle}$, then (X,d) is a metric space.

Exercise 1.2 (Cauchy-Schwarz Inequality). Let $(X,\langle\cdot,\cdot\rangle)$ be a complex inner product space. Let $x,y\in X$. Then

$$|\langle x,y\rangle|\leq\langle x,x\rangle^{1/2}\langle y,y\rangle^{1/2}.$$

Exercise 1.3. Let $(X,\langle\cdot,\cdot\rangle)$ be a complex inner product space. Let $x,y\in X$. As usual, let $\|x\|:=\sqrt{\langle x,x\rangle}$. Prove **Pythagoras's theorem**: if $\langle x,y\rangle=0$, then $\|x+y\|^2=\|x\|^2+\|y\|^2$.

Theorem 1.4 (Weierstrass M-test). *Let (X,d) be a metric space and let $(f_j)_{j=1}^{\infty}$ be a sequence of bounded real-valued continuous functions on X such that the series (of real numbers) $\sum_{j=1}^{\infty}\|f_j\|_\infty$ is absolutely convergent. Then the series $\sum_{j=1}^{\infty}f_j$ converges uniformly to some continuous function $f\colon X\to\mathbb{R}$.*

2. Introduction

A general problem in analysis is to approximate a general function by a series that is relatively easy to describe. With the Weierstrass Approximation theorem, we saw that it is possible to achieve this goal by approximating compactly supported continuous functions by polynomials. The notion of approximation here uses the sup norm. After our discussion of general series of functions, we focused on power series. In this case, real analytic functions can be written exactly in terms of their power series expansions. However, power series do not provide the best approximations for general functions.

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There is a different notion of approximation of general functions which we will now discuss. We will focus on periodic functions, and we will try to approximate these functions by trigonometric polynomials. As before, there are many choices of metrics in which we can say how close the approximating function is to the original function. These issues will be dealt with below, in our discussion of Fourier series. The topic of Fourier analysis can occupy more than one course, so we only select the introductory parts in this course.

3. Periodic Functions

Fourier series begins with the analysis of complex-valued, periodic functions.

Definition 3.1. Let $L > 0$ be a real number. A function $f: \mathbb{R} \to \mathbb{C}$ is **periodic with period** L, or $L-$**periodic**, if and only if $f(x+L) = f(x)$ for every real number x.

Example 3.2. The functions $\sin(x)$, $\cos(x)$ and e^{ix} are all 2π-periodic. They are also 4π-periodic, 6π-periodic, and so on. The function $f(x) = x$ is not periodic. The constant function $f(x) = 1$ is L-periodic for every $L > 0$.

Remark 3.3. If a function $f: \mathbb{R} \to \mathbb{C}$ is L-periodic, then $f(x + kL) = f(x)$ for every integer k. In particular, if f is 1-periodic, then $f(x + k) = f(x)$ for every integer k. So, 1-periodic functions are sometimes called $\mathbb{Z}$-**periodic functions** (and L-periodic functions are sometimes called $L\mathbb{Z}$-periodic functions.)

Example 3.4. For any integer n, the functions $\sin(2\pi nx)$, $\cos(2\pi nx)$ and $e^{2\pi inx}$ are $\mathbb{Z}$-periodic. For another example, consider the function where $f(x) = 1$ when $x \in [n, n + 1/2)$ for any integer n, and $f(x) = -1$ when $x \in [n + 1/2, n + 1)$ for any integer n. This function is an example of a **square wave**.

Remark 3.5. For simplicity, we will only deal with $\mathbb{Z}$-periodic functions below. The theory of general L-periodic functions follows relatively easily once the $\mathbb{Z}$-periodic theory has been developed. Note that a $\mathbb{Z}$-periodic function f is entirely determined by its values on the interval $[0, 1)$, since any $x \in \mathbb{R}$ can be written as $x = k + y$ where $k \in \mathbb{Z}$ and $y \in [0, 1)$, so that $f(x) = f(k + y) = f(y)$. Consequently, we sometimes describe a $\mathbb{Z}$-periodic function f by defining the function f on $[0, 1)$, and we then say that f is **extended periodically** by setting $f(k + y) = f(x) := f(y)$. (As before, we write $x \in \mathbb{R}$ as $x = k + y$ where $k \in \mathbb{Z}$ and $y \in [0, 1)$.)

The space of continuous complex-valued $\mathbb{Z}$-periodic functions is denoted by $C(\mathbb{R}/\mathbb{Z}; \mathbb{C})$. The notation $\mathbb{R}/\mathbb{Z}$ comes from algebra, where we consider the quotient of the additive group $\mathbb{R}$ by the additive group $\mathbb{Z}$. When we say that f is continuous and $\mathbb{Z}$-periodic, we mean that f is continuous on all of $\mathbb{R}$. If f is only continuous on the interval $[0, 1]$, then f may have a discontinuity at 0, so f may not be in $C(\mathbb{R}/\mathbb{Z}; \mathbb{C})$. For any integer n, the functions $\sin(2\pi nx)$, $\cos(2\pi nx)$ and $e^{2\pi inx}$ are in $C(\mathbb{R}/\mathbb{Z}; \mathbb{C})$. However, the square wave is not in $C(\mathbb{R}/\mathbb{Z}; \mathbb{C})$. Also, $\sin(x)$ is not in $C(\mathbb{R}/\mathbb{Z}; \mathbb{C})$ since it is not $\mathbb{Z}$-periodic.

Lemma 3.6.

(a) *(Continuous periodic functions are bounded.) If $f \in C(\mathbb{R}/\mathbb{Z}; \mathbb{C})$, then f is bounded. (That is, given $f \in C(\mathbb{R}/\mathbb{Z}; \mathbb{C})$, there exists $M > 0$ such that $|f(x)| \leq M$ for all $x \in \mathbb{R}$.)*

(b) *(Continuous periodic functions form a vector space and an algebra.)* *Let* $f, g \in C(\mathbb{R}/\mathbb{Z};\mathbb{C})$*. Then* $f+g$*,* $f-g$ *and* fg *are all in* $C(\mathbb{R}/\mathbb{Z};\mathbb{C})$*. Also, if* $c \in \mathbb{C}$*, then* $cf \in C(\mathbb{R}/\mathbb{Z};\mathbb{C})$*.*

(c) *(Uniform limits of continuous periodic functions are continuous periodic.) Let* $(f_j)_{j=1}^{\infty}$ *be a sequence of functions in* $C(\mathbb{R}/\mathbb{Z};\mathbb{C})$ *which converges uniformly to a function* $f\colon \mathbb{R} \to \mathbb{C}$*. Then* $f \in C(\mathbb{R}/\mathbb{Z};\mathbb{C})$*.*

Exercise 3.7. Prove Lemma 3.6. (Hint: for (i), first show that f is bounded on $[0,1]$.)

Remark 3.8. $C(\mathbb{R}/\mathbb{Z};\mathbb{C})$ becomes a metric space by re-introducing the sup-norm metric. Let $f, g \in C(\mathbb{R}/\mathbb{Z};\mathbb{C})$, and define

$$d_\infty(f,g) := \sup_{x\in\mathbb{R}} |f(x)-g(x)| = \sup_{x\in[0,1)} |f(x)-g(x)|.$$

In fact, $C(\mathbb{R}/\mathbb{Z};\mathbb{C})$ is a normed linear space with the norm

$$\|f\|_\infty := \sup_{x\in\mathbb{R}} |f(x)| = \sup_{x\in[0,1)} |f(x)|.$$

One can also show that $C(\mathbb{R}/\mathbb{Z};\mathbb{C})$ is a complete metric space.

4. Inner Products on Periodic Functions

We just discussed how to make $C(\mathbb{R}/\mathbb{Z};\mathbb{C})$ a normed linear space. We can also realize $\mathbb{C}(\mathbb{R}/\mathbb{Z};\mathbb{C})$ as a complex inner product space. However, the norm that is induced by this inner product will be *different* then the sup-norm. As we have mentioned above, there are many different norms in which we deal with functions. In this particular case, the most natural norm will not be the sup-norm. Instead, we will see that the norm that comes from the inner product will be more natural. We will discuss this issue further below, but for now we begin by defining the inner product on $C(\mathbb{R}/\mathbb{Z};\mathbb{C})$.

Definition 4.1. Let $f, g \in C(\mathbb{R}/\mathbb{Z};\mathbb{C})$. We define the (complex) **inner product** $\langle f, g\rangle$ to be the quantity

$$\langle f, g\rangle := \int_0^1 f(x)\overline{g(x)}dx.$$

Exercise 4.2. Verify that this inner product on $C(\mathbb{R}/\mathbb{Z};\mathbb{C})$ satisfies the axioms of a complex inner product space.

Remark 4.3. In order to integrate a general complex-valued function of the form $f(x) = g(x) + ih(x)$ where $h(x), g(x) \in \mathbb{R}$ for all $x \in [a,b]$, we define $\int_a^b f := (\int_a^b g) + i(\int_a^b h)$. For example,

$$\int_0^1 (1+ix) = 1 + i(\int_0^1 xdx) = 1 + i/2.$$

One can verify that all standard rules of calculus (integration by parts, the fundamental theorem of calculus, substitution, etc.) still hold when the function is complex-valued instead of real-valued.

Example 4.4. Let $f, g \in C(\mathbb{R}/\mathbb{Z};\mathbb{C})$ be the functions $f(x) = 1$ and $g(x) = e^{2\pi i x}$, for all $x \in \mathbb{R}$. Then

$$\langle f, g\rangle = \int_0^1 \overline{e^{2\pi i x}}dx = \int_0^1 e^{-2\pi i x}dx = \frac{e^{-2\pi i x}}{-2\pi i}\Big|_{x=0}^{x=1} = \frac{e^{-2\pi i}-1}{-2\pi i} = 0.$$

Remark 4.5. Let $f, g \in C(\mathbb{R}/\mathbb{Z};\mathbb{C})$. In general, $\langle f, g\rangle$ will be a complex number. Note also that since f, g are bounded and continuous, the function $f\overline{g}$ is Riemann integrable.

Definition 4.6. Let $f, g \in C(\mathbb{R}/\mathbb{Z};\mathbb{C})$. From Exercise 4.2, we see that $C(\mathbb{R}/\mathbb{Z};\mathbb{C})$ is a complex inner product space when equipped with the inner product $\langle f, g\rangle = \int_0^1 f\overline{g}$. So, from Exercise 1.1, we recall that this inner product makes $C(\mathbb{R}/\mathbb{Z};\mathbb{C})$ a normed linear space. We refer to this norm $\|f\|_2$ as the **L_2-norm** of f:

$$\|f\|_2 := \sqrt{\langle f, f\rangle} = \left(\int_0^1 f(x)\overline{f(x)}dx\right)^{1/2} = \left(\int_0^1 |f(x)|^2\, dx\right)^{1/2}.$$

The norm $\|f\|_2$ is sometimes called the **root mean square** of f.

Example 4.7. Let $f(x) := e^{2\pi i x}$. Then

$$\|f\|_2 = (\int_0^1 e^{2\pi i x}e^{-2\pi i x}dx)^{1/2} = (1)^{1/2} = 1.$$

Exercise 4.8. Let $M > 0$ be any positive real number. Find a function $f \in C(\mathbb{R}/\mathbb{Z};\mathbb{C})$ such that $\|f\|_2 \leq 1$ but such that $\|f\|_\infty > M$. On the other hand, if $g \in C(\mathbb{R}/\mathbb{Z};\mathbb{C})$, show that $\|g\|_2 \leq \|g\|_\infty$. So, the L_2 and sup-norms on $C(\mathbb{R}/\mathbb{Z};\mathbb{C})$ are related, but they can also be very different.

Definition 4.9. Due to Pythagoras's Theorem (Exercise 1.3), if $f, g \in C(\mathbb{R}/\mathbb{Z};\mathbb{C})$ satisfy $\langle f, g\rangle = 0$, we sometimes say that f, g are **orthogonal**.

Definition 4.10. Let $f, g \in C(\mathbb{R}/\mathbb{Z};\mathbb{C})$. From Exercise 1.1, we recall that the inner product on $C(\mathbb{R}/\mathbb{Z};\mathbb{C})$ also gives a metric on $C(\mathbb{R}/\mathbb{Z};\mathbb{C})$. We refer to this metric d_{L_2} as the L_2-metric:

$$d_{L_2}(f, g) := \sqrt{\langle (f-g), (f-g)\rangle} = \|f-g\|_2 = \left(\int_0^1 |f(x)-g(x)|^2\, dx\right)^{1/2}.$$

The L_2 metric shares many characteristics with the ℓ_2-metric on $\mathbb{R}^n$.

Remark 4.11. A sequence of functions $(f_j)_{j=1}^\infty$ in $C(\mathbb{R}/\mathbb{Z};\mathbb{C})$ will **converge in the L_2 metric** to $f \in C(\mathbb{R}/\mathbb{Z};\mathbb{C})$ if and only if $d_{L_2}(f_j, f) \to 0$ as $j \to \infty$. Equivalently,

$$\lim_{j\to\infty} \int_0^1 |f_j(x) - f(x)|^2\, dx = 0.$$

As we now show, convergence in the L_2 metric is different than both uniform convergence and pointwise convergence.

Exercise 4.12. Let $(f_j)_{j=1}^\infty$ be a sequence of functions in $C(\mathbb{R}/\mathbb{Z};\mathbb{C})$, and let f be another function in $C(\mathbb{R}/\mathbb{Z};\mathbb{C})$.

- Show that if $(f_j)_{j=1}^\infty$ converges uniformly to f, then $(f_j)_{j=1}^\infty$ also converges to f in the L_2 metric.
- Find a sequence $(f_j)_{j=1}^\infty$ in $C(\mathbb{R}/\mathbb{Z};\mathbb{C})$ which converges to some $f \in C(\mathbb{R}/\mathbb{Z};\mathbb{C})$ in the L_2 metric, so that $(f_j)_{j=1}^\infty$ does not converge to f uniformly. (Hint: consider $f = 0$ and use Exercise 4.8.)
- Find a sequence $(f_j)_{j=1}^\infty$ in $C(\mathbb{R}/\mathbb{Z};\mathbb{C})$ which converges to some $f \in C(\mathbb{R}/\mathbb{Z};\mathbb{C})$ in the L_2 metric, so that $(f_j)_{j=1}^\infty$ does not converge pointwise to f. (Hint: consider $f = 0$ and try to make the functions f_j large at one point.)

- Find a sequence $(f_j)_{j=1}^{\infty}$ in $C(\mathbb{R}/\mathbb{Z};\mathbb{C})$ which converges to some $f \in C(\mathbb{R}/\mathbb{Z};\mathbb{C})$ pointwise, so that $(f_j)_{j=1}^{\infty}$ does not converge to f in the L_2 metric. (Hint: consider $f = 0$ and try to make the functions f_j large in L_2 norm.)

Remark 4.13. Even though $C(\mathbb{R}/\mathbb{Z};\mathbb{C})$ is complete with respect to the sup-norm metric, it turns out that $C(\mathbb{R}/\mathbb{Z};\mathbb{C})$ is not complete with respect to the L_2 metric. For example, try to find a sequence of continuous functions that converges to the (discontinuous) square wave function.

5. Trigonometric Polynomials

In the theory of power series, we approximated functions by linear combinations of the monomials x^n where n is a positive integer. Now, in our discussion of Fourier series, we will approximate functions by linear combinations of the functions $e^{2\pi inx}$ where $n \in \mathbb{Z}$. The functions $e^{2\pi inx}$ are sometimes called **characters**.

To keep some simplicity in our notation, we make the following definition.

Definition 5.1. For every integer n, let $e_n \in C(\mathbb{R}/\mathbb{Z};\mathbb{C})$ denote the function

$$e_n(x) := e^{2\pi inx}, \quad x \in \mathbb{R}.$$

We sometimes refer to e_n as the **character with frequency** n.

Definition 5.2. A function $f \in C(\mathbb{R}/\mathbb{Z};\mathbb{C})$ is said to be a **trigonometric polynomial** if and only if there exists an integer $N \geq 0$ and there exists a sequence of complex numbers $(c_n)_{n=-N}^{N}$ such that

$$f = \sum_{n=-N}^{N} c_n e_n.$$

Example 5.3. The function $f = 2e_{-2} + 1 + 3e_1$ is a trigonometric polynomial. More explicitly, for all $x \in \mathbb{R}$, we have

$$f(x) = 2e^{-4\pi ix} + 1 + 3e^{2\pi ix}.$$

Example 5.4. For any integer n, the function $\cos(2\pi nx)$ is a trigonometric polynomial, since $\cos(2\pi nx) = (1/2)(e^{2\pi inx} + e^{-2\pi inx})$. Similarly, for any integer n, the function $\sin(2\pi nx)$ is a trigonometric polynomial, since $\sin(2\pi nx) = (1/(2i))(e_n(x) - e_{-n}(x))$. In particular, any linear combination of sines and cosines of this form is a trigonometric polynomial. For example, $\sin(4\pi x) + 3i\cos(2\pi x)$ is a trigonometric polynomial.

Remark 5.5. It turns out that *any* function in $C(\mathbb{R}/\mathbb{Z};\mathbb{C})$ can be written as an infinite sum of characters. That is, any function in $C(\mathbb{R}/\mathbb{Z};\mathbb{C})$ can be written as its Fourier series. The analogous statement for power series is that any real analytic function is equal to its power series.

The key fact used in proving this statement is given by the following computation.

Lemma 5.6 (Characters are an Orthonormal System). *Let n, m be integers. If $n = m$, then $\langle e_n, e_m\rangle = 1$. If $n \neq m$, then $\langle e_n, e_m\rangle = 0$. Also, $\|e_n\|_2 = 1$.*

Exercise 5.7. Prove Lemma 5.6.

Consequently, there is a nice formula to find the coefficients of a trigonometric polynomial.

Corollary 5.8. *Let $f = \sum_{n=-N}^{N} c_n e_n$ be a trigonometric polynomial. Then, for all integers $-N \leq n \leq N$, we have*

$$c_n = \langle f, e_n \rangle.$$

Also, for any integer n with $|n| > N$, we have $\langle f, e_n \rangle = 0$. And, we have the identity

$$\|f\|_2^2 = \sum_{n=-N}^{N} |c_n|^2.$$

Exercise 5.9. Prove Corollary 5.8. (Hint: for the final identity, use either the Pythagorean Theorem and induction, or substitute $f = \sum_{n=-N}^{N} c_n e_n$ into $\|f\|_2^2$ and expand out all of the terms.)

Definition 5.10. Let $f \in C(\mathbb{R}/\mathbb{Z}; \mathbb{C})$, and let $n \in \mathbb{Z}$. We define the n^{th} **Fourier coefficient of** f, denoted $\widehat{f}(n)$, to be the complex number

$$\widehat{f}(n) := \langle f, e_n \rangle = \int_0^1 f(x) e^{-2\pi i n x} dx.$$

The function $\widehat{f} \colon \mathbb{Z} \to \mathbb{C}$ is called the **Fourier transform** of f.

We now restate Corollary 5.8. From Corollary 5.8, whenever $f = \sum_{n=-N}^{N} c_n e_n$ is a trigonometric polynomial, we have

$$f = \sum_{n=-N}^{N} \langle f, e_n \rangle e_n = \sum_{n=-\infty}^{\infty} \langle f, e_n \rangle e_n.$$

That is, we have the **Fourier inversion formula**

$$f = \sum_{n=-\infty}^{\infty} \widehat{f}(n) e_n.$$

Put another way, for all $x \in \mathbb{R}$,

$$f(x) = \sum_{n=-\infty}^{\infty} \widehat{f}(n) e^{2\pi i n x}.$$

Also, from the second part of Corollary 5.8, we have **Plancherel's formula**

$$\|f\|_2^2 = \sum_{n=-\infty}^{\infty} |\widehat{f}(n)|^2.$$

Note that even though we have written these sums as infinite sums, they are actually finite sums, so there is no issue talking about their convergence. Below, we will extend the Fourier inversion and Plancherel formulas to general functions in $C(\mathbb{R}/\mathbb{Z}; \mathbb{C})$. These formulas hold on even larger classes of functions, but we may not have time to elaborate on this point. To prove the Fourier inversion formula, we will need a version of the Weierstrass approximation theorem for trigonometric polynomials. This proof will be analogous to the proof of the previous case of Weierstrass approximation (which we omitted).

6. Periodic Convolutions

In this section we will prove the following theorem.

Theorem 6.1 (**Weierstrass approximation theorem for trigonometric polynomials**)**.** *Let $f \in C(\mathbb{R}/\mathbb{Z};\mathbb{C})$ and let $\varepsilon > 0$. Then there exists a trigonometric polynomial P such that $\|f - P\|_\infty < \varepsilon$.*

In other words, any continuous periodic function can be uniformly approximated by trigonometric polynomials. In other words, if we let $P(\mathbb{R}/\mathbb{Z};\mathbb{C})$ denote the space of all trigonometric polynomials, then the closure of $P(\mathbb{R}/\mathbb{Z};\mathbb{C})$ in the L_∞ metric is $C(\mathbb{R}/\mathbb{Z};\mathbb{C})$.

The main tool in proving the Weierstrass approximation theorem is convolution.

Definition 6.2 (**Convolution**)**.** Let $f, g \in C(\mathbb{R}/\mathbb{Z};\mathbb{C})$. Then we define the **periodic convolution** $f * g \colon \mathbb{R} \to \mathbb{C}$ of f and g by the formula

$$f * g(x) := \int_0^1 f(y)g(x-y)dy, \qquad x \in \mathbb{R}.$$

For a fixed $x \in \mathbb{R}$, we can consider $f * g(x)$ to be a shifted average of the values of f and g. Below, we will see that the convolution can also be understood by looking at the Fourier transform of $f * g$.

Lemma 6.3 (**Properties of Convolution**)**.** *Let $f, g, h \in C(\mathbb{R}/\mathbb{Z};\mathbb{C})$.*

(a) *The convolution $f * g$ is continuous and $\mathbb{Z}$-periodic. That is, $f * g \in C(\mathbb{R}/\mathbb{Z};\mathbb{C})$.*
(b) *$f * g = g * f$.*
(c) *$f * (g + h) = f * g + f * h$ and $(f + g) * h = f * g + g * h$. For any complex number c, we have $c(f * g) = (cf) * g = f * (cg)$.*

Exercise 6.4. Prove Lemma 6.3. (Hints: to prove (a), you may need to use the uniform continuity of f and the boundedness of g, or vice versa. To prove $f * g = g * f$, you may need to use periodicity to "cut and paste" the interval $[0, 1]$.)

Let $f \in C(\mathbb{R}/\mathbb{Z};\mathbb{C})$ and let $n \in \mathbb{Z}$. Then

$$f * e_n = \widehat{f}(n)e_n.$$

Indeed, note that

$$\begin{aligned} f * e_n &= \int_0^1 f(y)e^{2\pi i n(x-y)}dy = e^{2\pi i n x}\int_0^1 f(y)e^{-2\pi i n y}dy \\ &= e^{2\pi i n x}\widehat{f}(n) = \widehat{f}(n)e_n. \end{aligned}$$

More generally, from Lemma 6.3(c), if $P = \sum_{n=-N}^{N} c_n e_n$ is any trigonometric polynomial, then

$$f * P = \sum_{n=-N}^{N} c_n(f * e_n) = \sum_{n=-N}^{N} \widehat{f}(n)c_n e_n \qquad (\ddagger)$$

In particular, we have the following

Lemma 6.5. *Let $f \in C(\mathbb{R}/\mathbb{Z};\mathbb{C})$ and let P be a trigonometric polynomial. Then $f * P$ is also a trigonometric polynomial.*

We can actually rewrite the equation (‡) as

$$\widehat{f * P}(n) = \widehat{f}(n)c_n = \widehat{f}(n)\widehat{P}(n).$$

In fact, even more generally, for any $f, g \in C(\mathbb{R}/\mathbb{Z};\mathbb{C})$, we have

$$\widehat{f * g}(n) = \widehat{f}(n)\widehat{g}(n). \qquad (\dagger)$$

In particular, even though the convolution can look a bit strange, we can instead interpret $f * g$ as simply being a function whose Fourier transform multiplies the Fourier transforms of f and g. The identity (†) is very important, though we will not use it this course.

Our strategy for proving the Weierstrass approximation theorem is the following. We will find a trigonometric polynomial P such that $f * P$ is close to f. From Lemma 6.5, we know that $f * P$ is a trigonometric polynomial. So, the strategy reduces to finding a trigonometric polynomial P such that $f * P$ is close to f. Since $f * P$ can be considered an average of the values of f and P, it turns out that we want to choose the polynomial P to be a positive function whose integral on $[0, 1]$ is mostly concentrated at a single point in $[0, 1]$. So, if P is very concentrated, then $f * P(x)$ will be mostly an average of the values of f near x. Then, the (uniform) continuity of f will guarantee that this average will be close to $f(x)$.

With this strategy in mind, we therefore look for a polynomial P that is positive and mostly concentrated at a single point. We call such a function an approximation of the identity.

Definition 6.6. Let $\varepsilon > 0$ and let $0 < \delta < 1/2$. A function $f \in C(\mathbb{R}/\mathbb{Z};\mathbb{C})$ is said to be an (ε, δ) **approximation of the identity** if and only if the following properties hold:

- $f(x) \geq 0$ for all $x \in \mathbb{R}$, and $\int_0^1 f = 1$.
- $f(x) < \varepsilon$ for all x with $\delta \leq |x| \leq 1 - \delta$.

Lemma 6.7. *For every $\varepsilon > 0$ and $0 < \delta < 1/2$, there exists a trigonometric polynomial P such that P is an (ε, δ) approximation of the identity.*

Proof. Let $N \geq 1$. We define the **Fejér kernel** F_N to be the function

$$F_N := \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N}\right) e_n.$$

Note that F_N is a trigonometric polynomial. Also,

$$\begin{aligned} F_N &= \frac{1}{N} \sum_{\ell=-N}^{N} (N - |\ell|)e_\ell = \frac{1}{N} \sum_{\ell=-N}^{N} \left(\sum_{-N+1 \leq j \leq 0 \leq k \leq N-1:\, j+k=\ell} e_\ell \right) \\ &= \frac{1}{N} \sum_{\ell=-N}^{N} \left(\sum_{-N+1 \leq j \leq 0 \leq k \leq N-1:\, j+k=\ell} e_j e_k \right) = \frac{1}{N} \sum_{k=0}^{N-1} e_k \overline{\Big(\sum_{j=0}^{N-1} e_j\Big)} = \frac{1}{N} \left| \sum_{n=0}^{N-1} e_n \right|^2. \end{aligned} \qquad (*)$$

And from the geometric series formula, for any real x that is not an integer,

$$\sum_{n=0}^{N-1} e_n(x) = \frac{e^{2\pi i N x} - 1}{e^{2\pi i x} - 1} = \frac{e^{\pi i N x}}{e^{\pi i x}} \cdot \frac{e^{\pi i N x} - e^{-\pi i N x}}{e^{\pi i x} - e^{-\pi i x}} = \frac{e^{\pi i (N-1) x} \sin(N\pi x)}{\sin(\pi x)}.$$

Combining this equation with $(*)$, we have for any real x that is not an integer,

$$F_N(x) = \frac{\sin^2(N\pi x)}{N\sin^2(\pi x)}. \qquad (**)$$

Also, when x is an integer, we see directly from $(*)$ that $F_N(x) = N$. So, $F_N(x) \geq 0$ for all real x. And by the definition of F_N and Lemma 5.6,

$$\int_0^1 F_N = \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N}\right) \int_0^1 e_n = (1 - 0/N)1 = 1.$$

Finally, since $|\sin(\pi N x)| \leq 1$, we conclude by $(**)$ that, if $\delta \leq |x| \leq 1-\delta$, then

$$F_N(x) \leq \frac{1}{N\sin^2(\pi x)} \leq \frac{1}{N\sin^2(\pi\delta)}.$$

The last inequality follows since sin is increasing on $[0, \pi/2]$, it is decreasing on $[\pi/2, \pi]$, and $\sin(\pi\delta) = \sin(-\pi\delta) = \sin(\pi(1-\delta))$, which uses that sin is odd and $\mathbb{Z}$-periodic. So, by choosing N large enough, we have $|F_N(x)| < \varepsilon$ for all x with $\delta \leq |x| \leq 1-\delta$. □

We can now prove the Weierstrass approximation theorem.

Proof of Theorem 6.1. Let $f \in C(\mathbb{R}/\mathbb{Z}; \mathbb{C})$. Then f is bounded, so there exists $M > 0$ such that $|f(x)| \leq M$ for all $x \in \mathbb{R}$.

Let $\varepsilon > 0$. Since f is uniformly continuous, there exists $\delta > 0$ such that, if $|x - y| < \delta$, then $|f(x) - f(y)| < \varepsilon$. By Lemma 6.7, let P be a trigonometric polynomial that is also an (ε, δ) approximation of the identity. Then $f * P$ is also a trigonometric polynomial by Lemma 6.5. So, it remains to show that $\|f - f * P\|_\infty < \varepsilon(2M + 2)$. Let $x \in \mathbb{R}$. Then

$$\begin{aligned}
|f(x) - f * P(x)| &= \left| f(x) - \int_0^1 P(y)f(x-y)dy \right| \\
&= \left| \int_0^1 P(y)f(x)dy - \int_0^1 P(y)f(x-y)dy \right| \qquad \text{, since } \int_0^1 P(y)dy = 1 \\
&= \left| \int_0^1 P(y)[f(x) - f(x-y)]dy \right| \\
&\leq \int_0^1 P(y)\,|f(x) - f(x-y)|\,dy \qquad \text{, using } P(y) \geq 0 \\
&= \int_0^\delta P(y)\,|f(x) - f(x-y)|\,dy + \int_\delta^{1-\delta} P(y)\,|f(x) - f(x-y)|\,dy \\
&\qquad + \int_{1-\delta}^1 P(y)\,|f(x) - f(x-y)|\,dy \\
&\leq \int_0^\delta P(y)\varepsilon dy + \int_\delta^{1-\delta} P(y)2M dy + \int_{1-\delta}^1 P(y)\,|f(x-1) - f(x-y)|\,dy \\
&\leq \varepsilon + 2M\varepsilon + \int_{1-\delta}^1 P(y)\varepsilon dy \leq \varepsilon(2M+2).
\end{aligned}$$

In conclusion, $\|f - f * P\|_\infty \leq \varepsilon(2M + 2)$. Since $\varepsilon > 0$ is arbitrary, we can find $f * P$ arbitrarily close to f in the sup norm, as desired. □

7. Fourier Inversion and Plancherel Theorems

Using the Weierstrass approximation (Theorem 6.1), we can now prove the Fourier inversion and Plancherel theorems for arbitrary functions in $C(\mathbb{R}/\mathbb{Z};\mathbb{C})$. The general theme here is that, Fourier inversion holds for any trigonometric polynomial, but trigonometric polynomials approximate functions in $C(\mathbb{R}/\mathbb{Z};\mathbb{C})$ arbitrarily well, so Fourier inversion also holds for functions in $C(\mathbb{R}/\mathbb{Z};\mathbb{C})$. Analogously, we know that a continuous function $F\colon \mathbb{R}\to\mathbb{R}$ which is zero on rational numbers is actually zero on all of $\mathbb{R}$.

Theorem 7.1 (**Fourier inversion/ Best approximation**). *For any $f\in C(\mathbb{R}/\mathbb{Z};\mathbb{C})$, the series $\sum_{n=-N}^{N}\widehat{f}(n)e_n$ converges to f in the L_2 metric. That is,*

$$\lim_{N\to\infty}\|f-\sum_{n=-N}^{N}\widehat{f}(n)e_n\|_2=0.$$

Proof. Let $\varepsilon>0$. We need N_0 such that, for all $N>N_0$, we have $\|f-\sum_{n=-N}^{N}\widehat{f}(n)e_n\|_2<\varepsilon$.

From the Weierstrass approximation theorem, there exists a natural number N_0 and there exists a trigonometric polynomial $P=\sum_{n=-N_0}^{N_0}c_ne_n$ such that $\|f-P\|_\infty<\varepsilon$. So, $\|f-P\|_2\leq\|f-P\|_\infty<\varepsilon$.

Let $N>N_0$, and let $f_N:=\sum_{n=-N}^{N}\widehat{f}(n)e_n$. We will conclude by showing that $\|f-f_N\|_2<\varepsilon$. First, note that, for any $m\in\mathbb{Z}$ with $|m|\leq N$, we have by Lemma 5.6

$$\langle f-f_N,e_m\rangle=\langle f,e_m\rangle-\sum_{n=-N}^{N}\widehat{f}(n)\langle e_n,e_m\rangle=\widehat{f}(m)-\widehat{f}(m)=0.$$

In particular, since f_N-P is a linear combination of e_m where $|m|\leq N$, we have

$$\langle f-f_N,f_N-P\rangle=0.$$

By the Pythagorean Theorem (Exercise 1.3), we therefore have

$$\|f-P\|_2^2=\|f-f_N\|_2^2+\|f_N-P\|_2^2.$$

Consequently,

$$\|f-f_N\|_2\leq\|f-P\|_2<\varepsilon.$$

□

Remark 7.2. Note that we have only proven convergence in the L_2 metric, and it is natural to look for other kinds of convergence. However, in general, f_N does not converge to f pointwise, and f_N does not converge to f uniformly. On the other hand, if we assume more about the function f, then we can get better convergence results. For example, if f is continuously differentiable, then f_N converges to f pointwise. And if f is twice continuously differentiable, then f_N converges to f uniformly. We will not cover these results here, and we instead defer them to the Fourier analysis course. Below, we only mention one theorem concerning uniform convergence.

Theorem 7.3. *Let $f \in C(\mathbb{R}/\mathbb{Z};\mathbb{C})$, and assume that the series (of numbers) $\sum_{n=-\infty}^{\infty} |\widehat{f}(n)|$ is absolutely convergent. Then the series $\sum_{n=-\infty}^{\infty} \widehat{f}(n)e_n$ converges uniformly to f. That is,*

$$\lim_{N\to\infty} \|f - \sum_{n=-N}^{N} \widehat{f}(n)e_n\|_\infty = 0.$$

Proof. By the Weierstrass M-test (Theorem 1.4), we know that $\sum_{n=-N}^{N} \widehat{f}(n)e_n$ converges uniformly to *some* function g. (Strictly speaking, the Weierstrass M-test applies to functions summed from $n = 0$ to $n = -\infty$, but this result applies to the situation at hand by splitting the double sum into two separate infinite sums.) By Lemma 3.6(c), g is continuous and periodic. So,

$$\lim_{N\to\infty} \|g - \sum_{n=-N}^{N} \widehat{f}(n)e_n\|_\infty = 0 \qquad (*).$$

Since the L_2 norm is bounded by the L_∞ norm, we conclude that

$$\lim_{N\to\infty} \|g - \sum_{n=-N}^{N} \widehat{f}(n)e_n\|_2 = 0.$$

By Theorem 7.1, we already know that

$$\lim_{N\to\infty} \|f - \sum_{n=-N}^{N} \widehat{f}(n)e_n\|_2 = 0.$$

By the uniqueness of limits in metric spaces (in this case, uniqueness of limits with respect to the L_2 metric), we conclude that $f = g$. That is, $(*)$ concludes the proof. □

As a Corollary of Fourier inversion, we obtain the following theorem.

Theorem 7.4 (Plancherel theorem). *Let $f \in C(\mathbb{R}/\mathbb{Z};\mathbb{C})$. Then the series (of numbers) $\sum_{n=-\infty}^{\infty} |\widehat{f}(n)|^2$ is absolutely convergent. Also,*

$$\|f\|_2^2 = \sum_{n=-\infty}^{\infty} |\widehat{f}(n)|^2.$$

Proof. Let $\varepsilon > 0$. By the Fourier inversion theorem (Theorem 7.1), there exists N_0 such that, for all $N > N_0$, we have

$$\|f - \sum_{n=-N}^{N} \widehat{f}(n)e_n\|_2 < \varepsilon.$$

So, by the triangle inequality,

$$\|f\|_2 - \varepsilon < \|\sum_{n=-N}^{N} \widehat{f}(n)e_n\|_2 \leq \|f\|_2 + \varepsilon.$$

Moreover, by Corollary 5.8, we have

$$\|\sum_{n=-N}^{N} \widehat{f}(n)e_n\|_2 = (\sum_{n=-N}^{N} |\widehat{f}(n)|^2)^{1/2}.$$

Therefore,

$$(\|f\|_2 - \varepsilon)^2 < \sum_{n=-N}^{N} |\widehat{f}(n)|^2 < (\|f\|_2 + \varepsilon)^2.$$

Taking the limit superior and limit inferior, we have

$$(\|f\|_2 - \varepsilon)^2 \leq \liminf_{N\to\infty} \sum_{n=-N}^{N} |\widehat{f}(n)|^2 \leq \limsup_{N\to\infty} \sum_{n=-N}^{N} |\widehat{f}(n)|^2 \leq (\|f\|_2 + \varepsilon)^2.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $\lim_{N\to\infty} \sum_{n=-N}^{N} |\widehat{f}(n)|^2 = \|f\|_2^2$, as desired. □

8. Appendix: Notation

Let A, B be sets in a space X. Let m, n be a nonnegative integers.

$$
\begin{aligned}
\mathbb{Z} &:= \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}, \text{ the integers} \\
\mathbb{N} &:= \{0, 1, 2, 3, 4, 5, \ldots\}, \text{ the natural numbers} \\
\mathbb{Z}_+ &:= \{1, 2, 3, 4, \ldots\}, \text{ the positive integers} \\
\mathbb{Q} &:= \{m/n \colon m, n \in \mathbb{Z}, n \neq 0\}, \text{ the rationals} \\
\mathbb{R} &\text{ denotes the set of real numbers} \\
\mathbb{R}^* &= \mathbb{R} \cup \{-\infty\} \cup \{+\infty\} \text{ denotes the set of extended real numbers} \\
\mathbb{C} &:= \{x + y\sqrt{-1} : x, y \in \mathbb{R}\}, \text{ the complex numbers} \\
\emptyset &\text{ denotes the empty set, the set consisting of zero elements} \\
\in &\text{ means "is an element of." For example, } 2 \in \mathbb{Z} \text{ is read as "2 is an element of } \mathbb{Z}\text{."} \\
\forall &\text{ means "for all"} \\
\exists &\text{ means "there exists"} \\
\mathbb{R}^n &:= \{(x_1, \ldots, x_n) \colon x_i \in \mathbb{R}, \, \forall \, i \in \{1, \ldots, n\}\} \\
A \subseteq B &\text{ means } \forall \, a \in A, \text{ we have } a \in B, \text{ so } A \text{ is contained in } B \\
A \smallsetminus B &:= \{x \in A \colon x \notin B\} \\
A^c &:= X \smallsetminus A, \text{ the complement of } A \\
A \cap B &\text{ denotes the intersection of } A \text{ and } B \\
A \cup B &\text{ denotes the union of } A \text{ and } B
\end{aligned}
$$

Let (X, d) be a metric space, let $x_0 \in X$, let $r > 0$ be a real number, and let E be a subset of X. Let $(x_1, \ldots, x_n)$ be an element of $\mathbb{R}^n$, and let $p \geq 1$ be a real number.

$$
\begin{aligned}
B_{(X,d)}(x_0, r) = B(x_0, r) &:= \{x \in X \colon d(x, x_0) < r\}. \\
\overline{E} &\text{ denotes the closure of } E \\
\mathrm{int}(E) &\text{ denotes the interior of } E \\
\partial E &\text{ denotes the boundary of } E
\end{aligned}
$$

$$
\begin{aligned}
\|(x_1, \ldots, x_n)\|_{\ell_p} &:= (\sum_{i=1}^{n} |x_i|^p)^{1/p} \\
\|(x_1, \ldots, x_n)\|_{\ell_\infty} &:= \max_{i=1,\ldots,n} |x_i|
\end{aligned}
$$

Let $f,g\colon (X,d_X)\to(Y,d_Y)$ be maps between metric spaces. Let $V\subseteq X$, and let $W\subseteq Y$.

$f(V):=\{f(v)\in Y\colon v\in V\}$.

$f^{-1}(W):=\{x\in X\colon f(x)\in W\}$.

$d_\infty(f,g):=\sup_{x\in X} d_Y(f(x),g(x))$.

$B(X;Y)$ denotes the set of functions $f\colon X\to Y$ that are bounded.

$C(X;Y):=\{f\in B(X;Y)\colon f \text{ is continuous}\}$.

Let $f,g\colon\mathbb{R}\to\mathbb{C}$ be $\mathbb{Z}$-periodic functions.

$$\|f\|_\infty := \sup_{x\in[0,1)} |f(x)|.$$

$$\langle f,g\rangle := (\int_0^1 f(x)\overline{g(x)}dx)^{1/2}.$$

$$\|f\|_2 := \sqrt{\langle f,f\rangle} = (\int_0^1 |f(x)|^2\, dx)^{1/2}$$

$$d_{L_2}(f,g) := \|f-g\|_2 = (\int_0^1 |f(x)-g(x)|^2\, dx)^{1/2}.$$

8.1. **Set Theory.** Let X,Y be sets, and let $f\colon X\to Y$ be a function. The function $f\colon X\to Y$ is said to be **injective** (or **one-to-one**) if and only if: for every $x,x'\in V$, if $f(x)=f(x')$, then $x=x'$.

The function $f\colon X\to Y$ is said to be **surjective** (or **onto**) if and only if: for every $y\in Y$, there exists $x\in X$ such that $f(x)=y$.

The function $f\colon X\to Y$ is said to be **bijective** (or a **one-to-one correspondence**) if and only if: for every $y\in Y$, there exists exactly one $x\in X$ such that $f(x)=y$. A function $f\colon X\to Y$ is bijective if and only if it is both injective and surjective.

Two sets X,Y are said to have the same **cardinality** if and only if there exists a bijection from X onto Y.

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