6: SEQUENCES AND SERIES OF FUNCTIONS, CONVERGENCE

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Contents

1. Review

Remark 1.1. From now on, unless otherwise specified, $\mathbb{R}^n$ refers to Euclidean space $\mathbb{R}^n$ with $n \geq 1$ a positive integer, and where we use the metric d_{ℓ_2} on $\mathbb{R}^n$. In particular, $\mathbb{R}$ refers to the metric space $\mathbb{R}$ equipped with the metric $d(x, y) = |x - y|$.

Proposition 1.2. *Let (X, d) be a metric space. Let $(x^{(j)})_{j=k}^{\infty}$ be a sequence of elements of X. Let x, x' be elements of X. Assume that the sequence $(x^{(j)})_{j=k}^{\infty}$ converges to x with respect to d. Assume also that the sequence $(x^{(j)})_{j=k}^{\infty}$ converges to x' with respect to d. Then $x = x'$.*

Proposition 1.3. *Let $a < b$ be real numbers, and let $f\colon [a, b] \to \mathbb{R}$ be a function which is both continuous and strictly monotone increasing. Then f is a bijection from $[a, b]$ to $[f(a), f(b)]$, and the inverse function $f^{-1}\colon [f(a), f(b)] \to [a, b]$ is also continuous and strictly monotone increasing.*

Theorem 1.4 (Inverse Function Theorem). *Let X, Y be subsets of $\mathbb{R}$. Let $f\colon X \to Y$ be bijection, so that $f^{-1}\colon Y \to X$ is a function. Let $x_0 \in X$ and $y_0 \in Y$ such that $f(x_0) = y_0$. If f is differentiable at x_0, if f^{-1} is continuous at y_0, and if $f'(x_0) \neq 0$, then f^{-1} is differentiable at y_0 with*

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)}.$$

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2. Sequences of Functions

As we have seen in analysis, it is often desirable to discuss sequences of points that converge. Below, we will see that it is similarly desirable to discuss sequences of functions that converge in various senses. There are many distinct ways of discussing the convergence of sequences of functions. We will only discuss two such modes of convergence, namely pointwise and uniform convergence. Before beginning this discussion, we discuss the limiting values of functions between metric spaces, which should generalize our notion of limiting values of functions on the real line.

2.1. Limiting Values of Functions.

Definition 2.1. Let (X, d_X) and (Y, d_Y) be metric spaces, let E be a subset of X, let $f\colon X \to Y$ be a function, let $x_0 \in X$ be an adherent point of E, and let $L \in Y$. We say that $f(x)$ **converges to** L **in** Y **as** x **converges to** x_0 **in** E, and we write $\lim_{x\to x_0; x\in E} f(x) = L$, if and only if, for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that, if $x \in E$ satisfies $d_X(x, x_0) < \delta$, then $d_Y(f(x), L) < \varepsilon$.

Remark 2.2. So, f is continuous at x_0 if and only if

$$\lim_{x\to x_0; x\in X} f(x) = f(x_0). \qquad (*)$$

And f is continuous on X if and only if, for all $x_0 \in X$, $(*)$ holds.

Remark 2.3. When the domain of x of the limit $\lim_{x\to x_0; x\in X} f(x)$ is clear, we will often instead write $\lim_{x\to x_0} f(x)$.

The following equivalence is generalized from its analogue on the real line.

Proposition 2.4. *Let (X, d_X) and (Y, d_Y) be metric spaces, let E be a subset of X, let $f\colon X \to Y$ be a function, let $x_0 \in X$ be an adherent point of E, and let $L \in Y$. Then the following statements are equivalent.*

- $\lim_{x\to x_0; x\in E} f(x) = L$.
- *For any sequence $(x^{(j)})_{j=1}^{\infty}$ in E which converges to x_0 with respect to the metric d_X, the sequence $(f(x^{(j)}))_{j=1}^{\infty}$ converges to L with respect to the metric d_Y.*

Exercise 2.5. Prove Proposition 2.4.

Remark 2.6. From Propositions 2.4 and 1.2, the function f can converge to at most one limit L as x converges to x_0.

Remark 2.7. The notation $\lim_{x\to x_0; x\in E} f(x)$ implicitly refers to a convergence of the function values $f(x)$ in the metric space (Y, d_Y). Strictly speaking, it would be better to write d_Y somewhere next to the notation $\lim_{x\to x_0; x\in E} f(x)$. However, this omission of notation should not cause confusion.

2.2. Pointwise Convergence and Uniform Convergence.

Definition 2.8 (**Pointwise Convergence**)**.** Let (X, d_X) and (Y, d_Y) be metric spaces. Let $(f_j)_{j=1}^{\infty}$ be a sequence of functions from X to Y. Let $f\colon X \to Y$ be another function. We say that $(f_j)_{j=1}^{\infty}$ **converges pointwise** to f on X if and only if, for every $x \in X$, we have

$$\lim_{j\to\infty} f_j(x) = f(x).$$

That is, for all $x \in X$, we have

$$\lim_{j\to\infty} d_Y(f_j(x), f(x)) = 0.$$

That is, for every $x \in X$ and for every $\varepsilon > 0$, there exists $J > 0$ such that, for all $j > J$, we have $d_Y(f_j(x), f(x)) < \varepsilon$.

Remark 2.9. Note that, if we change the point x, then the limiting behavior of $f_j(x)$ can change quite a bit. For example, let j be a positive integer, and consider the functions $f_j\colon [0,1] \to \mathbb{R}$ where $f_j(x) = j$ for all $x \in (0, 1/j)$, and $f_j(x) = 0$ otherwise. Let $f\colon [0,1] \to \mathbb{R}$ be the zero function. Then f_j converges pointwise to zero, since for any $x \in (0,1]$, we have $f_j(x) = 0$ for all $j > 1/x$. (And $f_j(0) = 0$ for all positive integers j.) However, given any fixed positive integer j, there exists an x such that $f_j(x) = j$. Moreover, $\int_0^1 f_j = 1$ for all positive integers j, but $\int_0^1 f = 0$. So, we see that pointwise convergence does not preserve the integral of a function.

Remark 2.10. Pointwise convergence also does not preserve continuity. For example, consider $f_j\colon [0,1] \to \mathbb{R}$ defined by $f_j(x) = x^j$, where $j \in \mathbb{N}$ and $x \in [0,1]$. Define $f\colon [0,1] \to \mathbb{R}$ so that $f(1) = 1$ and so that $f(x) = 0$ for $x \in [0,1)$. Then f_j converges pointwise to f as $j \to \infty$, and each f_j is continuous, but f is not continuous.

In summary, pointwise convergence doesn't really preserve any useful analytic quantities. The above remarks show that some points are changing at much different rates than other points as $j \to \infty$. A stronger notion of convergence will then fix these issues, where all points in the domain are controlled simultaneously.

Definition 2.11 (Uniform Convergence). Let (X, d_X) and (Y, d_Y) be metric spaces. Let $(f_j)_{j=1}^{\infty}$ be a sequence of functions from X to Y. Let $f\colon X \to Y$ be another function. We say that $(f_j)_{j=1}^{\infty}$ **converges uniformly** to f on X if and only if, for every $\varepsilon > 0$, there exists $J > 0$ such that, for all $j > J$ and for all $x \in X$ we have $d_Y(f_j(x), f(x)) < \varepsilon$.

Remark 2.12. Note that the difference between uniform and pointwise convergence is that we simply moved the quantifier "for all $x \in X$" within the statement. This change means that the integer J does not depend on x in the case of uniform convergence.

Remark 2.13. The sequences of functions from Remarks 2.9 and 2.10 do not converge uniformly. So, pointwise convergence does not imply uniform convergence. However, uniform convergence does imply pointwise convergence.

3. Uniform Convergence and Continuity

We saw that pointwise convergence does not preserve continuity. However, uniform convergence does preserve continuity.

Theorem 3.1. *Let (X, d_X) and (Y, d_Y) be metric spaces. Let $(f_j)_{j=1}^{\infty}$ be a sequence of functions from X to Y. Let $f\colon X \to Y$ be another function. Let $x_0 \in X$. Suppose f_j converges uniformly to f on X. Suppose that, for each $j \geq 1$, we know that f_j is continuous at x_0. Then f is also continuous at x_0.*

Exercise 3.2. Prove Theorem 3.1. Hint: it is probably easiest to use the $\varepsilon - \delta$ definition of continuity. Once you do this, you may require the triangle inequality in the form

$$d_Y(f(x), f(x_0)) \leq d_Y(f(x), f_j(x)) + d_Y(f_j(x), f_j(x_0)) + d_Y(f_j(x_0), f(x_0)).$$

Corollary 3.3. *Let (X, d_X) and (Y, d_Y) be metric spaces. Let $(f_j)_{j=1}^{\infty}$ be a sequence of functions from X to Y. Let $f: X \to Y$ be another function. Suppose $(f_j)_{j=1}^{\infty}$ converges uniformly to f on X. Suppose that, for each $j \geq 1$, we know that f_j is continuous on X. Then f is also continuous on X.*

Uniform limits of bounded functions are also bounded. Recall that a function $f: X \to Y$ between metric spaces (X, d_X) and (Y, d_Y) is **bounded** if and only if there exists a radius $R > 0$ and a point $y_0 \in Y$ such that $f(x) \in B_{(Y,d_Y)}(y_0, R)$ for all $x \in X$.

Proposition 3.4. *Let (X, d_X) and (Y, d_Y) be metric spaces. Let $(f_j)_{j=1}^{\infty}$ be a sequence of functions from X to Y. Let $f: X \to Y$ be another function. Suppose $(f_j)_{j=1}^{\infty}$ converges uniformly to f on X. Suppose also that, for each $j \geq 1$, we know that f_j is bounded. Then f is also bounded.*

Exercise 3.5. Prove Proposition 3.4.

3.1. **The Metric of Uniform Convergence.** We will now see one advantage to our abstract approach to analysis on metric spaces. We can in fact talk about uniform convergence in terms of a metric on a space of functions, as follows.

Definition 3.6. Let (X, d_X) and (Y, d_Y) be metric spaces. Let $B(X;Y)$ denote the set of functions $f: X \to Y$ that are bounded. Let $f, g \in B(X;Y)$. We define the metric $d_\infty: B(X;Y) \times B(X;Y) \to [0, \infty)$ by

$$d_\infty(f, g) := \sup_{x \in X} d_Y(f(x), g(x)).$$

This metric is known as the **sup norm metric** or the L_∞ **metric**. We also use $d_{B(X;Y)}$ as a synonym for d_∞. Note that $d_\infty(f, g) < \infty$ since f, g are assumed to be bounded.

Exercise 3.7. Show that the space $(B(X;Y), d_\infty)$ is a metric space.

Example 3.8. Let $X = [0, 1]$ and let $Y = \mathbb{R}$. Consider the functions $f(x) = x$ and $g(x) = 2x$ where $x \in [0, 1]$. Then f, g are bounded, and

$$d_\infty(f, g) = \sup_{x \in [0,1]} |x - 2x| = \sup_{x \in [0,1]} |x| = 1.$$

Here is our promised characterization of uniform convergence in terms of the metric d_∞.

Proposition 3.9. *Let (X, d_X) and (Y, d_Y) be metric spaces. Let $(f_j)_{j=1}^{\infty}$ be a sequence of functions in $B(X;Y)$. Let $f \in B(X;Y)$. Then $(f_j)_{j=1}^{\infty}$ converges uniformly to f on X if and only if $(f_j)_{j=1}^{\infty}$ converges to f in the metric $d_{B(X;Y)}$.*

Exercise 3.10. Prove Proposition 3.9.

Definition 3.11. Let (X, d_X) and (Y, d_Y) be metric spaces. Define the set of bounded continuous functions from X to Y as

$$C(X;Y) := \{f \in B(X;Y) : f \text{ is continuous}\}.$$

Note that $C(X;Y) \subseteq B(X;Y)$ by the definition of $C(X;Y)$. Also, by Corollary 3.3, $C(X;Y)$ is closed in $B(X;Y)$ with respect to the metric d_∞. In fact, more is true.

Theorem 3.12. *Let (X, d_X) be a metric space, and let (Y, d_Y) be a complete metric space. Then the space $(C(X;Y), d_{B(X;Y)}|_{C(X;Y)\times C(X;Y)})$ is a complete subspace of $B(X;Y)$. That is, every Cauchy sequence of functions in $C(X;Y)$ converges to a function in $C(X;Y)$.*

Exercise 3.13. Prove Theorem 3.12

4. Series of Functions and the Weierstrass M-test

For each positive integer j, let $f_j\colon X \to \mathbb{R}$ be a function. We will now consider infinite series of the form $\sum_{j=1}^{\infty} f_j$. The most natural thing to do now is to determine in what sense the series $\sum_{j=1}^{\infty} f_j$ is a function, and if it is a function, determine if it is continuous. Note that we have restricted the range to be $\mathbb{R}$ since it does not make sense to add elements in a general metric space. Power series and Fourier series perhaps give the most studied examples of series of functions. If $x \in [0,1]$ and if a_j are real numbers for all $j \geq 1$, we want to make sense of the series $\sum_{j=1}^{\infty} a_j \cos(2\pi jx)$. We want to know in what sense this infinite series is a function, and if it is a function, do the partial sums converge in any reasonable manner? We will return to these issues later on.

Definition 4.1. Let (X, d_X) be a metric space. For each positive integer j, let $f_j\colon X \to \mathbb{R}$ be a function, and let $f\colon X \to \mathbb{R}$ be another function. If the partial sums $\sum_{j=1}^{J} f_j$ converge pointwise to f as $J \to \infty$, then we say that the infinite series $\sum_{j=1}^{\infty} f_j$ **converge pointwise** to f, and we write $f = \sum_{j=1}^{\infty} f_j$. If the partial sums $\sum_{j=1}^{J} f_j$ converge uniformly to f as $J \to \infty$, then we say that the infinite series $\sum_{j=1}^{\infty} f_j$ **converge uniformly** to f, and we write $f = \sum_{j=1}^{\infty} f_j$. (In particular, the notation $f = \sum_{j=1}^{\infty} f_j$ is ambiguous, since the nature of the convergence of the series is not specified.)

Remark 4.2. If a series converges uniformly then it converges pointwise. However, the converse is false in general.

Exercise 4.3. Let $x \in (-1,1)$. For each integer $j \geq 1$, define $f_j(x) := x^j$. Show that the series $\sum_{j=1}^{\infty} f_j$ converges pointwise, but not uniformly, on $(-1,1)$ to the function $f(x) = x/(1-x)$. Also, for any $0 < t < 1$, show that the series $\sum_{j=1}^{\infty} f_j$ converges uniformly to f on $[-t,t]$.

Definition 4.4. Let $f\colon X \to \mathbb{R}$ be a bounded real-valued function. We define the **sup-norm** $\|f\|_\infty$ of f to be the real number

$$\|f\|_\infty := \sup_{x\in X} |f(x)|.$$

Exercise 4.5. Let X be a set. Show that $\|\cdot\|_\infty$ is a norm on the space $B(X;\mathbb{R})$.

Theorem 4.6 (**Weierstrass M-test**)**.** *Let (X, d) be a metric space and let $(f_j)_{j=1}^{\infty}$ be a sequence of bounded real-valued continuous functions on X such that the series (of real numbers) $\sum_{j=1}^{\infty} \|f_j\|_\infty$ is absolutely convergent. Then the series $\sum_{j=1}^{\infty} f_j$ converges uniformly to some continuous function $f\colon X \to \mathbb{R}$.*

Exercise 4.7. Prove Theorem 4.6. (Hint: first, show that the partial sums $\sum_{j=1}^{J} f_j$ form a Cauchy sequence in $C(X;\mathbb{R})$. Then, use Theorem 3.12 and the completeness of the real line $\mathbb{R}$.)

Remark 4.8. The Weierstrass M-test will be useful in our investigation of power series.

5. Uniform Convergence and Integration

Theorem 5.1. *Let $a < b$ be real numbers. For each integer $j \geq 1$, let $f_j\colon [a,b] \to \mathbb{R}$ be a Riemann integrable function on $[a,b]$. Suppose f_j converges uniformly on $[a,b]$ to a function $f\colon [a,b] \to \mathbb{R}$, as $j \to \infty$. Then f is also Riemann integrable, and*

$$\lim_{j\to\infty} \int_a^b f_j = \int_a^b f.$$

Remark 5.2. Before we begin, recall that we require any Riemann integrable function g to be bounded. Also, for a Riemann integrable function g, we denote $\underline{\int_a^b} g$ as the supremum of all lower Riemann sums of g over all partitions of $[a,b]$. And we denote $\overline{\int_a^b} g$ as the infimum of all upper Riemann sums of g over all partitions of $[a,b]$. Recall also that a function g is defined to be Riemann integrable if and only if $\underline{\int_a^b} g = \overline{\int_a^b} g$.

Proof. We first show that f is Riemann integrable. First, note that f_j is bounded for all $j \geq 1$, since that is part of the definition of being Riemann integrable. So, f is bounded by Proposition 3.4. Now, let $\varepsilon > 0$. Since f_j converges uniformly to f on $[a,b]$, there exists $J > 0$ such that, for all $j > J$, we have

$$f_j(x) - \varepsilon \leq f(x) \leq f_j(x) + \varepsilon, \qquad \forall\, x \in [a,b].$$

Integrating this inequality on $[a,b]$, we have

$$\underline{\int_a^b} (f_j(x) - \varepsilon) \leq \underline{\int_a^b} f \leq \overline{\int_a^b} f \leq \overline{\int_a^b} (f_j(x) + \varepsilon).$$

Since f_j is Riemann integrable for all $j \geq 1$, we therefore have

$$-(b-a)\varepsilon + \int_a^b f_j \leq \underline{\int_a^b} f \leq \overline{\int_a^b} f \leq (b-a)\varepsilon + \int_a^b f_j. \qquad (*)$$

In particular, we get

$$0 \leq \overline{\int_a^b} f - \underline{\int_a^b} f \leq 2(b-a)\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $\overline{\int_a^b} f = \underline{\int_a^b} f$, so f is Riemann integrable.

Now, from $(*)$, we have: for any $\varepsilon > 0$, there exists J such that, for all $j > J$, we have

$$\left| \int_a^b f - \int_a^b f_j \right| \leq (b-a)\varepsilon.$$

Since this holds for any $\varepsilon > 0$, we conclude that $\lim_{j\to\infty} \int_a^b f_j = \int_a^b f$, as desired. □

Remark 5.3. In summary, if a sequence of Riemann integrable functions $(f_j)_{j=1}^{\infty}$ converges to f uniformly, then we can interchange limits and integrals

$$\lim_{j\to\infty}\int f_j = \int \lim_{j\to\infty} f_j.$$

Recall that this equality does not hold if we only assume that the functions converge pointwise.

An analogous statement holds for series.

Theorem 5.4. *Let $a < b$ be real numbers. For each integer $j \geq 1$, let $f_j\colon [a,b] \to \mathbb{R}$ be a Riemann integrable function on $[a,b]$. Suppose $\sum_{j=1}^{\infty} f_j$ converges uniformly on $[a,b]$. Then $\sum_{j=1}^{\infty} f_j$ is also Riemann integrable, and*

$$\sum_{j=1}^{\infty}\int_a^b f_j = \int_a^b \sum_{j=1}^{\infty} f_j.$$

Exercise 5.5. Prove Theorem 5.4.

Example 5.6. Let $x \in (-1,1)$. We know that $\sum_{j=1}^{\infty} x^j = x/(1-x)$, and the convergence is uniform on $[-r,r]$ for any $0 < r < 1$. Adding 1 to both sides, we get

$$\sum_{j=0}^{\infty} x^j = \frac{1}{1-x}.$$

And this sum also converges uniformly on $[-r,r]$ for any $0 < r < 1$. Applying Theorem 5.4 and integrating on $[0,r]$, we get

$$\sum_{j=0}^{\infty}\frac{r^{j+1}}{j+1} = \sum_{j=0}^{\infty}\int_0^r x^j = \int_0^r \frac{1}{1-x}.$$

The last function is equal to $-\log(1-r)$, though we technically have not defined the logarithm function yet. We will define the logarithm further below.

6. Uniform Convergence and Differentiation

We now investigate the relation between uniform convergence and differentiation.

Remark 6.1. Suppose a sequence of differentiable functions $(f_j)_{j=1}^{\infty}$ converges uniformly to a function f. We first show that f need not be differentiable. Consider the functions $f_j(x) := \sqrt{x^2 + 1/j}$, where $x \in [-1,1]$. Let $f(x) = |x|$. Note that

$$|x| \leq \sqrt{x^2 + 1/j} \leq |x| + 1/\sqrt{j}.$$

These inequalities follow by taking the square root of $x^2 \leq x^2 + 1/j \leq x^2 + 1/j + 2\,|x|\,/\sqrt{j}$. So, by the Squeeze Theorem, $(f_j)_{j=1}^{\infty}$ converges uniformly to f on $[-1,1]$. However, f is not differentiable at 0. In conclusion, uniform convergence does not preserve differentiability.

Remark 6.2. Suppose a sequence of differentiable functions $(f_j)_{j=1}^{\infty}$ converge uniformly to a function f. Even if f is assumed to be differentiable, we show that $(f_j)'$ may not converge to f'. Consider the functions $f_j(x) := j^{-1/2}\sin(j\pi x)$, where $x \in [-1,1]$. (We will assume some basic properties of trigonometric functions which we will prove later on. Since we are only providing a motivating example, we will not introduce any circular reasoning.) Let f be the zero function. Since $|\sin(j\pi x)| \leq 1$, we have $d_\infty(f_j, f) \leq j^{-1/2}$, so $(f_j)_{j=1}^\infty$ converges uniformly on $[-1,1]$. However, $f_j'(x) = j^{1/2}\pi\cos(j\pi x)$. So, $f_j'(0) = j^{1/2}\pi$. That is, $(f_j')_{j=1}^\infty$ does not converge pointwise to f. So, $(f_j')_{j=1}^\infty$ does not converge uniformly to $f' = 0$. In conclusion, uniform convergence does not imply uniform convergence of derivatives.

However, the converse statement is true, as long as the sequence of functions converges at one point.

Theorem 6.3. *Let $a < b$. For every integer $j \geq 1$, let $f_j \colon [a,b] \to \mathbb{R}$ be a differentiable function whose derivative $(f_j)' \colon [a,b] \to \mathbb{R}$ is continuous. Assume that the derivatives $(f_j)'$ converge uniformly to a function $g \colon [a,b] \to \mathbb{R}$ as $j \to \infty$. Assume also that there exists a point $x_0 \in [a,b]$ such that $\lim_{j\to\infty} f_j(x_0)$ exists. Then the functions f_j converge uniformly to a differentiable function f as $j \to \infty$, and $f' = g$.*

Proof. Let $x \in [a,b]$. From the Fundamental Theorem of Calculus, for each $j \geq 1$,

$$f_j(x) - f_j(x_0) = \int_{x_0}^{x} f_j'. \qquad (*)$$

By assumption, $L := \lim_{j\to\infty} f_j(x_0)$ exists. From Theorem 3.1, g is continuous, and in particular, g is Riemann integrable on $[a,b]$. Also, by Theorem 5.1, $\lim_{j\to\infty}\int_{x_0}^{x} f_j'$ exists and is equal to $\int_{x_0}^{x} g$. We conclude by $(*)$ that $\lim_{j\to\infty} f_j(x)$ exists, and

$$\lim_{j\to\infty} f_j(x) = L + \int_{x_0}^{x} g.$$

Define the function f on $[a,b]$ so that

$$f(x) = L + \int_{x_0}^{x} g.$$

We know so far that $(f_j)_{j=1}^\infty$ converges pointwise to f. We now need to show that this convergence is in fact uniform. We defer this part to the exercises. □

Exercise 6.4. Complete the proof of Theorem 6.3.

Corollary 6.5. *Let $a < b$. For every integer $j \geq 1$, let $f_j \colon [a,b] \to \mathbb{R}$ be a differentiable function whose derivative $f_j' \colon [a,b] \to \mathbb{R}$ is continuous. Assume that the series of real numbers $\sum_{j=1}^\infty \|f_j'\|_\infty$ is absolutely convergent. Assume also that there exists $x_0 \in [a,b]$ such that the series of real numbers $\sum_{j=1}^\infty f_j(x_0)$ converges. Then the series $\sum_{j=1}^\infty f_j$ converges uniformly on $[a,b]$ to a differentiable function. Moreover, for all $x \in [a,b]$,*

$$\frac{d}{dx}\sum_{j=1}^{\infty} f_j(x) = \sum_{j=1}^{\infty}\frac{d}{dx} f_j(x)$$

Exercise 6.6. Prove Corollary 6.5.

The following exercise is a nice counterexample to keep in mind, and it also shows the necessity of the assumptions of Corollary 6.5.

Exercise 6.7. (For this exercise, you can freely use facts about trigonometry that you learned in your previous courses.) Let $x \in \mathbb{R}$ and let $f\colon \mathbb{R} \to \mathbb{R}$ be the function $f(x) := \sum_{j=1}^{\infty} 4^{-j}\cos(32^j \pi x)$. Note that this series is uniformly convergent by the Weierstrass M-test (Theorem 4.6). So, f is a continuous function. However, at every point $x \in \mathbb{R}$, f is not differentiable, as we now discuss.

- Show that, for all positive integers j, m, we have
$$|f((j+1)/32^m) - f(j/32^m)| \geq 4^{-m}.$$
(Hint: for certain sequences of numbers $(a_j)_{j=1}^{\infty}$, use the identity
$$\sum_{j=1}^{\infty} a_j = (\sum_{j=1}^{m-1} a_j) + a_m + \sum_{j=m+1}^{\infty} a_j.$$
Also, use the fact that the cosine function is periodic with period 2π, and the summation $\sum_{j=0}^{\infty} r^j = 1/(1-r)$ for all $-1 < r < 1$. Finally, you should require the inequality: for all real numbers x, y, we have $|\cos(x) - \cos(y)| \leq |x - y|$. This inequality follows from the Mean Value Theorem or the Fundamental Theorem of Calculus.)
- Using the previous result, show that, for ever $x \in \mathbb{R}$, f is not differentiable at x. (Hint: for every $x \in \mathbb{R}$ and for every positive integer m, there exists an integer j such that $j \leq 32^m x \leq j+1$.)
- Explain briefly why this result does not contradict Corollary 6.5.

7. Uniform Approximation by Polynomials

Definition 7.1 (Polynomial). Let $a < b$ be real numbers and let $x \in [a, b]$. A **polynomial** on $[a, b]$ is a function $f\colon [a, b] \to \mathbb{R}$ of the form $f(x) = \sum_{j=0}^{k} a_j x^j$, where k is a natural number and $a_0, \ldots, a_k$ are real numbers. If $a_k \neq 0$, then k is called the **degree of** f.

From the previous exercise, we have seen that general continuous functions can behave rather poorly, in that they may never be differentiable. Polynomials on the other hand are infinitely differentiable. And it is often beneficial to deal with polynomials instead of general functions. So, we mention below a result of Weierstrass which says: any continuous function on an interval $[a, b]$ can be uniformly approximated by polynomials.

This fact seems to be related to power series, but it is something much different. It may seem possible to take a general (infinitely differentiable) function, take a high degree Taylor polynomial of this function, and then claim that this polynomial approximates our original function well. There are two problems with this approach. First of all, the continuous function that we start with may not even be differentiable. Second of all, even if we have an infinitely differentiable function, its power series may not actually approximate that function well. Recall that the function $f(x) = e^{-1/x^2}$ (where $f(0) := 0$) is infinitely differentiable, but its Taylor polynomial is identically zero at $x = 0$. In conclusion, we need to use something other than Taylor series to approximate a general continuous function by polynomials.

The proof of the Weierstrass approximation theorem introduces several useful ideas, but it is typically only proven in the honors class. However, later on, we will prove a version of this theorem for trigonometric polynomials, and this proof will be analogous to the proof of the current theorem.

Theorem 7.2 (**Weierstrass approximation**). *Let $a < b$ be real numbers. Let $f\colon [a,b] \to \mathbb{R}$ be a continuous function, and let $\varepsilon > 0$. Then there exists a polynomial P on $[a,b]$ such that $d_\infty(P,f) < \varepsilon$. (That is, $|f(x) - P(x)| < \varepsilon$ for all $x \in [a,b]$.)*

Remark 7.3. We can also state this Theorem using metric space terminology. Recall that $C([a,b];\mathbb{R})$ is the space of continuous functions from $[a,b]$ to $\mathbb{R}$, equipped with the sup-norm metric d_∞. Let $P([a,b];\mathbb{R})$ be the space of all polynomials on $[a,b]$, so that $P([a,b];\mathbb{R})$ is a subspace of $C([a,b];\mathbb{R})$, since polynomials are continuous. Then the Weierstrass approximation theorem says that every continuous function is an adherent point of $P([a,b];\mathbb{R})$. Put another way, the closure of $P([a,b];\mathbb{R})$ is $C([a,b];\mathbb{R})$.

$$\overline{P([a,b];\mathbb{R})} = C([a,b];\mathbb{R}).$$

Put another way, every continuous function on $[a,b]$ is the uniform limit of polynomials.

8. Power Series

We now focus our discussion of series to power series.

Definition 8.1 (**Power Series**). Let a be a real number, let $(a_j)_{j=0}^\infty$ be a sequence of real numbers, and let $x \in \mathbb{R}$. A **formal power series centered at** a is a series of the form

$$\sum_{j=0}^{\infty} a_j (x-a)^j,$$

For a natural number j, we refer to a_j as the j^{th} **coefficient** of the power series.

Remark 8.2. We refer to these power series as formal since their convergence is not guaranteed. Note however that any formal power series centered at a converges at $x = a$. It turns out that we can precisely identify where a formal power series converges just from the asymptotic behavior of the coefficients.

Definition 8.3 (**Radius of Convergence**). Let $\sum_{j=0}^\infty a_j(x-a)^j$ be a formal power series. The **radius of convergence** $R \geq 0$ of this series is defined to be

$$R := \frac{1}{\limsup_{j\to\infty} |a_j|^{1/j}}.$$

In the definition of R, we use the convention that $1/0 = +\infty$ and $1/(+\infty) = 0$. Note that it is possible for R to then take any value between and including 0 and $+\infty$. Note also that R always exists as a nonnegative real number, or as $+\infty$, since the limit superior of a positive sequence always exists as a nonegative number, or $+\infty$.

Example 8.4. The radius of convergence of the series $\sum_{j=0}^\infty j(-2)^j(x-3)^j$ is

$$\frac{1}{\limsup_{j\to\infty} |j(-2)^j|^{1/j}} = \frac{1}{\limsup_{j\to\infty} 2j^{1/j}} = \frac{1}{2}.$$

The radius of convergence of the series $\sum_{j=0}^\infty 2^{j^2}(x+2)^j$ is

$$\frac{1}{\limsup_{j\to\infty} 2^j} = \frac{1}{+\infty} = 0.$$

The radius of convergence of the series $\sum_{j=0}^{\infty} 2^{-j^2}(x+2)^j$ is

$$\frac{1}{\limsup_{j\to\infty} 2^{-j}} = \frac{1}{0} = +\infty.$$

As we now show, the radius of convergence tells us exactly where the power series converges.

Theorem 8.5. *Let $\sum_{j=0}^{\infty} a_j(x-a)^j$ be a formal power series, and let R be its radius of convergence.*

(a) *(Divergence outside of the radius of convergence) If $x \in \mathbb{R}$ satisfies $|x-a| > R$, then the series $\sum_{j=0}^{\infty} a_j(x-a)^j$ is divergent at x.*
(b) *(Convergence inside the radius of convergence) If $x \in \mathbb{R}$ satisfies $|x-a| < R$, then the series $\sum_{j=0}^{\infty} a_j(x-a)^j$ is convergent at x.*
- *For the following items (c),(d) and (e), we assume that $R > 0$. Then, let $f\colon (a-R, a+R)$ be the function $f(x) = \sum_{j=0}^{\infty} a_j(x-a)^j$, which exists by part (b).*

(c) *(Uniform convergence on compact intervals) For any $0 < r < R$, we know that the series $\sum_{j=0}^{\infty} a_j(x-a)^j$ converges uniformly to f on $[a-r, a+r]$. In particular, f is continuous on $(a-R, a+R)$ (by Theorem 3.1.)*
(d) *(Differentiation of power series) The function f is differentiable on $(a-R, a+R)$. For any $0 < r < R$, the series $\sum_{j=0}^{\infty} ja_j(x-a)^{j-1}$ converges uniformly to f' on the interval $[a-r, a+r]$.*
(e) *(Integration of power series) For any closed interval $[y, z]$ contained in $(a-R, a+R)$, we have*

$$\int_y^z f = \sum_{j=0}^{\infty} a_j \frac{(z-a)^{j+1} - (y-a)^{j+1}}{j+1}.$$

Exercise 8.6. Prove Theorem 8.5. (Hints: for parts (a),(b), use the root test. For part (c), use the Weierstrass M-test. For part (d), use Theorem 6.3. For part (e), use Theorem 5.4.)

Remark 8.7. A power series may converge or diverge when $|x-a| = R$.

Exercise 8.8. Give examples of formal power series centered at 0 with radius of convergence $R = 1$ such that

- The series diverges at $x = 1$ and at $x = -1$.
- The series diverges at $x = 1$ and converges at $x = -1$.
- The series converges at $x = 1$ and diverges at $x = 1$.
- The series converges at $x = 1$ and at $x = -1$.

We now discuss functions that are equal to convergent power series.

Definition 8.9. Let $a \in \mathbb{R}$ and let $r > 0$. Let E be a subset of $\mathbb{R}$ such that $(a-r, a+r) \subseteq E$. Let $f\colon E \to \mathbb{R}$. We say that the function f is **real analytic on** $(a-r, a+r)$ if and only if there exists a power series $\sum_{j=0}^{\infty} a_j(x-a)^j$ centered at a with radius of convergence R such that $R \geq r$ and such that this power series converges to f on $(a-r, a+r)$.

Example 8.10. The function $f\colon (0,2) \to \mathbb{R}$ defined by $f(x) = \sum_{j=0}^{\infty} j(x-1)^j$ is real analytic on $(0, 2)$.

From Theorem 8.5, if a function f is real analytic on $(a-r, a+r)$, then f is continuous and differentiable. In fact, f is can be differentiated any number of times, as we now show.

Definition 8.11. Let E be a subset of $\mathbb{R}$. We say that a function $f\colon E \to \mathbb{R}$ is **once differentiable on** E if and only if f is differentiable on E. More generally, for any integer $k \geq 2$, we say that $f\colon E \to \mathbb{R}$ is k **times differentiable on** E, or just k **times differentiable**, if and only if f is differentiable and f' is $k-1$ times differentiable. If f is k times differentiable, we define the k^{th} derivative $f^{(k)}\colon E \to \mathbb{R}$ by the recursive rule $f^{(1)} := f'$ and $f^{(k)} := (f^{(k-1)})'$, for all $k \geq 2$. We also define $f^{(0)} := f$. A function is said to be **infinitely differentiable** if and only if f is k times differentiable for every $k \geq 0$.

Example 8.12. The function $f(x) = |x|^3$ is twice differentiable on $\mathbb{R}$, but not three times differentiable on $\mathbb{R}$. Note that $f''(x) = 6\,|x|$, which is not differentiable at $x = 0$.

Proposition 8.13. *Let $a \in \mathbb{R}$ and let $r > 0$. Let f be a function that is real analytic on $(a-r, a+r)$, with the power series expansion*

$$f(x) = \sum_{j=0}^{\infty} a_j (x-a)^j, \quad \forall\, x \in (a-r, a+r).$$

Then, for any integer $k \geq 0$, the function f is k times differentiable on $(a-r, a+r)$, and the k^{th} derivative is given by

$$f^{(k)}(x) = \sum_{j=0}^{\infty} a_{j+k}(j+1)(j+2)\cdots(j+k)(x-a)^j, \quad \forall\, x \in (a-r, a+r).$$

Exercise 8.14. Prove Proposition 8.13.

Corollary 8.15 (Taylor's formula). *Let $a \in \mathbb{R}$ and let $r > 0$. Let f be a function that is real analytic on $(a-r, a+r)$, with the power series expansion*

$$f(x) = \sum_{j=0}^{\infty} a_j (x-a)^j, \quad \forall\, x \in (a-r, a+r).$$

Then, for any integer $k \geq 0$, we have

$$f^{(k)}(a) = k! a_k,$$

*where $k! = 1 \times 2 \times \cdots \times k$, and we denote $0! := 1$. In particular, we have **Taylor's formula***

$$f(x) = \sum_{j=0}^{\infty} \frac{f^{(j)}(a)}{j!}(x-a)^j, \quad \forall\, x \in (a-r, a+r).$$

Exercise 8.16. Prove Corollary 8.15 using Proposition 8.13.

Remark 8.17. The series $\sum_{j=0}^{\infty} \frac{f^{(j)}(a)}{j!}(x-a)^j$ is sometimes called the **Taylor series** of f around a. Taylor's formula says that if f is real analytic, then f is equal to its Taylor series. In the following exercise, we see that even if f is infinitely differentiable, it may not be equal to its Taylor series.

Exercise 8.18. Define a function $f\colon \mathbb{R} \to \mathbb{R}$ by $f(0) := 0$ and $f(x) := e^{-1/x^2}$ for $x \neq 0$. Show that f is infinitely differentiable, but $f^{(k)}(0) = 0$ for all $k \geq 0$. So, being infinitely differentiable does not imply that f is equal to its Taylor series. (You may freely use properties of the exponential function that you have learned before.)

Corollary 8.19 (**Uniqueness of power series**)**.** *Let $a \in \mathbb{R}$ and let $r > 0$. Let f be a function that is real analytic on $(a-r, a+r)$, with two power series expansions*

$$f(x) = \sum_{j=0}^{\infty} a_j (x-a)^j, \quad \forall\, x \in (a-r, a+r).$$

$$f(x) = \sum_{j=0}^{\infty} b_j (x-a)^j, \quad \forall\, x \in (a-r, a+r).$$

Then $a_j = b_j$ for all $j \geq 0$.

Proof. By Corollary 8.15, we have $k!a_k = f^{(k)}(a) = k!b_k$ for all $k \geq 0$. Since $k! \neq 0$ for all $k \geq 0$, we divide by $k!$ to get $a_k = b_k$ for all $k \geq 0$. $\square$

Remark 8.20. Note however that a power series can have very different expansions if we change the center of the expansion. For example, the function $f(x) = 1/(1-x)$ satisfies

$$f(x) = \sum_{j=0}^{\infty} x^j, \quad \forall\, x \in (-1, 1).$$

However, at the point $1/2$, we have the different expansion

$$f(x) = \frac{1}{1-x} = \frac{2}{1-2(x-1/2)} = \sum_{j=0}^{\infty} 2(2(x-1/2))^j = \sum_{j=0}^{\infty} 2^{j+1}(x-1/2)^j, \quad \forall x \in (0,1).$$

Note also that the first series has radius of convergence 1 and the second series has radius of convergence $1/2$.

8.1. **Multiplication of Power Series.**

Lemma 8.21 (**Fubini's Theorem for Series**)**.** *Let $f\colon \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ be a function such that $\sum_{(j,k) \in \mathbb{N} \times \mathbb{N}} f(j,k)$ is absolutely convergent. (That is, for any bijection $g\colon \mathbb{N} \to \mathbb{N} \times \mathbb{N}$, the sum $\sum_{\ell=0}^{\infty} f(g(\ell))$ is absolutely convergent.) Then*

$$\sum_{j=1}^{\infty} (\sum_{k=1}^{\infty} f(j,k)) = \sum_{(j,k) \in \mathbb{N} \times \mathbb{N}} f(j,k) = \sum_{k=1}^{\infty} (\sum_{j=1}^{\infty} f(j,k)).$$

Proof Sketch. We only consider the case $f(j,k) \geq 0$ for all $(j,k) \in \mathbb{N}$. The general case then follows by writing $f = \max(f,0) - \min(f,0)$, and applying this special case to $\max(f,0)$ and $\min(f,0)$, separately.

Let $L := \sum_{(j,k) \in \mathbb{N} \times \mathbb{N}} f(j,k)$. For any $J, K > 0$, we have $\sum_{j=1}^{J} \sum_{k=1}^{K} f(j,k) \leq L$. Letting $J, K \to \infty$, we conclude that $\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} f(j,k) \leq L$. Let $\varepsilon > 0$. It remains to find J, K such that $\sum_{j=1}^{J} \sum_{k=1}^{K} > L - \varepsilon$. Since $\sum_{(j,k) \in \mathbb{N} \times \mathbb{N}} f(j,k)$ converges absolutely, there exists a finite set $X \subseteq \mathbb{N} \times \mathbb{N}$ such that $\sum_{(j,k) \in X} f(j,k) > L - \varepsilon$. But then we can choose J, K sufficiently large such that $\{(j,k) \in X\} \subseteq \{(j,k) \colon 1 \leq j \leq J, 1 \leq k \leq K\}$. Therefore, $\sum_{j=1}^{J} \sum_{k=1}^{K} f(j,k) \geq \sum_{(j,k) \in X} f(j,k) > L - \varepsilon$, as desired. $\square$

Theorem 8.22. *Let $a \in \mathbb{R}$ and let $r > 0$. Let f and g be functions that are real analytic on $(a-r, a+r)$, with power series expansions*

$$f(x) = \sum_{j=0}^{\infty} a_j (x-a)^j, \quad \forall\, x \in (a-r, a+r).$$

$$g(x) = \sum_{j=0}^{\infty} b_j (x-a)^j, \quad \forall\, x \in (a-r, a+r).$$

Then the function fg is also real analytic on $(a-r, a+r)$. For each $j \geq 0$, define $c_j := \sum_{k=0}^{j} a_k b_{j-k}$. Then fg has the power series expansion

$$f(x)g(x) = \sum_{j=0}^{\infty} c_j (x-a)^j, \quad \forall\, x \in (a-r, a+r).$$

Proof. Fix $x \in (a-r, a+r)$. By Theorem 8.5, both f and g have radius of convergence $R \geq r$. So, both $\sum_{j=0}^{\infty} a_j (x-a)^j$ and $\sum_{j=0}^{\infty} b_j (x-a)^j$ are absolutely convergent. Define

$$C := \sum_{j=0}^{\infty} \left| a_j (x-a)^j \right|, \qquad D := \sum_{j=0}^{\infty} \left| b_j (x-a)^j \right|.$$

Then both C, D are finite.

For any $N \geq 0$, consider the partial sum

$$\sum_{j=0}^{N} \sum_{k=0}^{N} \left| a_j (x-a)^j b_k (x-a)^k \right|.$$

We can re-write this sum as

$$\sum_{j=0}^{N} \left| a_j (x-a)^j \right| \sum_{k=0}^{N} \left| b_k (x-a)^k \right| \leq \sum_{j=0}^{N} \left| a_j (x-a)^j \right| D \leq CD.$$

Since this inequality holds for all $N \geq 0$, the series

$$\sum_{(j,k) \in \mathbb{N} \times \mathbb{N}} \left| a_j (x-a)^j b_k (x-a)^k \right|$$

is convergent. That is, the following series is absolutely convergent.

$$\sum_{(j,k) \in \mathbb{N} \times \mathbb{N}} a_j (x-a)^j b_k (x-a)^k.$$

Now, using Lemma 8.21,

$$\sum_{(j,k) \in \mathbb{N} \times \mathbb{N}} a_j (x-a)^j b_k (x-a)^k = \sum_{j=0}^{\infty} a_j (x-a)^j \sum_{k=0}^{\infty} b_k (x-a)^k = \sum_{j=0}^{\infty} a_j (x-a)^j g(x) = f(x)g(x).$$

Rewriting this equality,

$$f(x)g(x) = \sum_{(j,k) \in \mathbb{N} \times \mathbb{N}} a_j (x-a)^j b_k (x-a)^k = \sum_{(j,k) \in \mathbb{N} \times \mathbb{N}} a_j b_k (x-a)^{j+k}.$$

Since the sum is absolutely convergent, we can rearrange the order of summation. For any fixed positive integer ℓ, we sum over all positive integers j, k such that $j + k = \ell$. That is, we have

$$f(x)g(x) = \sum_{\ell=0}^{\infty} \sum_{(j,k)\in\mathbb{N}\times\mathbb{N}:\, j+k=\ell} a_j b_k (x-a)^{\ell} = \sum_{\ell=0}^{\infty} (x-a)^{\ell} \sum_{s=0}^{\ell} a_s b_{s-\ell}.$$

□

9. The Exponential and Logarithm

We can now use the material from the previous sections to define and investigate various special functions.

Definition 9.1. For every real number x, we define the **exponential function** $\exp(x)$ to be the real number

$$\exp(x) := \sum_{j=0}^{\infty} \frac{x^j}{j!}.$$

Theorem 9.2 (Properties of the Exponential Function).

(a) *For every real number x, the series $\sum_{j=0}^{\infty} \frac{x^j}{j!}$ is absolutely convergent. So,* $\exp(x)$ *exists and is a real number for every $x \in \mathbb{R}$, the power series $\sum_{j=0}^{\infty} \frac{x^j}{j!}$ has radius of convergence $R = +\infty$, and* $\exp$ *is an analytic function on $(-\infty, +\infty)$.*
(b) $\exp$ *is differentiable on $\mathbb{R}$, and for every $x \in \mathbb{R}$, we have* $\exp'(x) = \exp(x)$.
(c) $\exp$ *is continuous on $\mathbb{R}$, and for all real numbers $a < b$, we have* $\int_a^b \exp = \exp(b) - \exp(a)$.
(d) *For every $x, y \in \mathbb{R}$, we have* $\exp(x + y) = \exp(x)\exp(y)$.
(e) $\exp(0) = 1$. *Also, for every $x \in \mathbb{R}$, we have* $\exp(x) > 0$, *and* $\exp(-x) = 1/\exp(x)$.
(f) $\exp$ *is strictly monotone increasing. That is, whenever x, y are real numbers with $x < y$, we have* $\exp(x) < \exp(y)$.

Exercise 9.3. Prove Theorem 9.2. (Hints: for part (a), use the ratio test. For parts (b) and (c), use Theorem 8.5. For part (d), you may need the binomial formula $(x + y)^k = \sum_{j=0}^{k} \frac{k!}{j!(k-j)!} x^j y^{k-j}$. For part (e), use part (d). For part (f), use part (d) and show that $\exp(x) > 1$ for all $x > 0$.)

Definition 9.4. We define the real number e by

$$e := \exp(1) = \sum_{j=0}^{\infty} \frac{1}{j!}$$

Proposition 9.5. *For every real number x, we have*

$$\exp(x) = e^x.$$

Exercise 9.6. Prove Proposition 9.5. (Hint: first prove the proposition for natural numbers x. Then, prove the proposition for integers. Then, prove the proposition for rational numbers. Finally, use the density of the rationals to prove the proposition for real numbers. You should find useful identifies for exponentiation by rational numbers.)

From now on, we use $\exp(x)$ and e^x interchangeably.

Remark 9.7. Since $e > 1$ by the definition of e, we have $e^x \to +\infty$ as $x \to +\infty$ and $e^x \to 0$ as $x \to -\infty$. So, from the Intermediate Value Theorem, the range of exp is $(0, \infty)$. Since exp is strictly increasing on $\mathbb{R}$, exp is therefore injective on $\mathbb{R}$, so exp is a bijection from $\mathbb{R}$ to $(0, \infty)$. Therefore, exp has an inverse function from $(0, \infty)$ to $\mathbb{R}$.

Definition 9.8. We define the **natural logarithm function** $\log\colon (0, \infty) \to \mathbb{R}$ (which is also called ln) to be the inverse of the exponential function. So, $\exp(\log(x)) = x$ for every $x \in (0, \infty)$, and $\log(\exp(x)) = x$ for every $x \in \mathbb{R}$.

Remark 9.9. Since exp is continuous and strictly monotone increasing, log is also continuous and strictly monotone increasing by Proposition 1.3. Since exp is differentiable and its derivative is never zero, the Inverse Function Theorem (Theorem 1.4) implies that log is also differentiable.

Theorem 9.10.

(a) *For every $x \in (0, \infty)$, we have $\log'(x) = 1/x$. So, by the Fundamental Theorem of Calculus, for any $0 < a < b$, we have $\int_a^b (1/t)dt = \log(b) - \log(a)$.*
(b) *For all $x, y \in (0, \infty)$, we have $\log(x) + \log(y) = \log(xy)$.*
(c) *For all $x \in (0, \infty)$, we have $\log(1/x) = -\log x$. In particular, $\log(1) = 0$.*
(d) *For any $x \in (0, \infty)$ and $y \in \mathbb{R}$, we have $\log(x^y) = y \log x$.*
(e) *For any $x \in (-1, 1)$, we have*

$$-\log(1 - x) = \sum_{j=1}^{\infty} \frac{x^j}{j}.$$

In particular, log is analytic on $(0, 2)$ with the power series expansion

$$\log(x) = \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j}(x - 1)^j, \quad \forall\, x \in (0, 2).$$

Exercise 9.11. Prove Theorem 9.10. (Hints: for part (a), use the Inverse Function Theorem or Chain Rule. For parts (b),(c) and (d), use Theorem 9.2 and the laws of exponentiation. For part (e), let $x \in (-1, 1)$, use the geometric series formula $1/(1 - x) = \sum_{j=0}^{\infty} x^j$ and integrate using Theorem 8.5.)

9.1. **A Digression concerning Complex Numbers.** Our investigation of trigonometric functions below is significantly improved by the introduction of the complex number system. We will also use the complex exponential in our discussion of Fourier series.

Definition 9.12 (Complex Numbers). A **complex number** is any expression of the form $a + bi$ where a, b are real numbers. Formally, the symbol i is a placeholder with no intrinsic meaning. Two complex numbers $a + bi$ and $c + di$ are said to be equal if and only if $a = c$ and $b = d$. Every real number x is considered a complex number, with the identification $x = x+0i$. The sum of two complex numbers is defined by $(a+bi)+(c+di) := (a+c)+(b+d)i$. The difference of two complex numbers is defined by $(a+bi)-(c+di) := (a-c)+(b-d)i$. The product of two complex numbers is defined by $(a+bi)(c+di) := (ac-bd)+(ad+bc)i$. If $c+di \neq 0$, the quotient of two complex numbers is defined by $(a+bi)/(c+di) := (a+bi)(\frac{c}{c^2+d^2} - \frac{d}{c^2+d^2}i)$.

The **complex conjugate** of a complex number $a+bi$ is defined by $\overline{a+bi} := a-bi$. The absolute value of a complex number $a+bi$ is defined by $|a+bi| := \sqrt{a^2+b^2}$. The space of all complex numbers is called $\mathbb{C}$.

Remark 9.13. We write i as shorthand for $0+i$. Note that $i^2=-1$.

Remark 9.14. The complex numbers obey all of the usual rules of algebra. For example, if v,w,z are complex numbers, then $v(w+z)=vw+vz$, $v(wz)=(vw)z$, and so on. Specifically, the complex numbers $\mathbb{C}$ form a **field**. Also, the rules of complex arithmetic are consistent with the rules of real arithmetic. That is, $3+5=8$ whether or not we use addition in $\mathbb{R}$ or addition in $\mathbb{C}$.

The operation of complex conjugation preserves all of the arithmetic operations. If w,z are complex numbers, then $\overline{w+z}=\overline{w}+\overline{z}$, $\overline{w-z}=\overline{w}-\overline{z}$, $\overline{w\cdot z}=\overline{w}\cdot\overline{z}$, and $\overline{w/z}=\overline{w}/\overline{z}$ for $z\neq 0$. The complex conjugate and absolute value satisfy $|z|^2=z\overline{z}$.

Remark 9.15. If $z\in\mathbb{C}$, then $|z|=0$ if and only if $z=0$. If $z,w\in\mathbb{C}$, then it can be shown that $|zw|=|z|\,|w|$, and if $w\neq 0$, then $|z/w|=|z|\,/\,|w|$. Also, the triangle inequality holds: $|z+w|\leq |z|+|w|$. So, $\mathbb{C}$ is a metric space if we use the metric $d(z,w):=|z-w|$. Moreover, $\mathbb{C}$ is a complete metric space.

The theory we have developed to deal with series of real functions also covers complex-valued functions, with almost no change to the proofs. For example, we can define the exponential function of a complex number z by

$$\exp(z) := \sum_{j=0}^{\infty}\frac{z^j}{j!}.$$

The ratio test then can be proven in exactly the same manner for complex series, and it follows that $\exp(z)$ converges for every $z\in\mathbb{C}$. Many of the properties of Theorem 9.2 still hold, though we cannot deal with all of these properties in this class. However, the following identity is proven in the exact same way as in the setting of real numbers: for any $z,w\in\mathbb{C}$, we have

$$\exp(z+w)=\exp(z)\exp(w).$$

Also, we should note that $\overline{\exp(z)}=\exp(\overline{z})$, which follows by conjugating the partial sums $\sum_{j=0}^{J} z^j/j!$, and then letting $J\to\infty$.

We briefly mention that the complex logarithm is more difficult to define, mainly because the exponential function is not invertible on $\mathbb{C}$. This topic is deferred to the complex analysis class.

10. Trigonometric Functions

Besides the exponential and logarithmic functions, there are many different kinds of special functions. Here, we will only mention the sine and cosine functions. One's first encounter with the sine and cosine functions probably involved their definition in terms of the edge lengths of right triangles. However, we will show below an analytic definition of these functions, which will also facilitate the investigation of the properties that they possess. The complex exponential plays a crucial role in this development.

Definition 10.1. Let x be a real number. We then define

$$\cos(x) := \frac{e^{ix} + e^{-ix}}{2}.$$
$$\sin(x) := \frac{e^{ix} - e^{-ix}}{2i}.$$

We refer to cos as the **cosine** function, and we refer to sin as the **sine** function.

Remark 10.2. Using the power series expansion for the exponential, we can then derive power series expansions for sine and cosine as follows. Let $x \in \mathbb{R}$. Then

$$e^{ix} = 1 + ix - x^2/2! - ix^3/3! + x^4/4! + \cdots$$
$$e^{-ix} = 1 - ix - x^2/2! + ix^3/3! + x^4/4! - \cdots$$

Therefore, using the definitions of sine and cosine,

$$\cos(x) = 1 - x^2/2! + x^4/4! - \cdots = \sum_{j=0}^{\infty} \frac{(-1)^j x^{2j}}{(2j)!}.$$
$$\sin(x) = x - x^3/3! + x^5/5! - \cdots = \sum_{j=0}^{\infty} \frac{(-1)^j x^{2j+1}}{(2j+1)!}.$$

So, if $x \in \mathbb{R}$ then $\cos(x) \in \mathbb{R}$ and $\sin(x) \in \mathbb{R}$. Also, sine and cosine real analytic on $(-\infty, \infty)$, e.g. since their power series converge on $(-\infty, \infty)$ by the ratio test. In particular, the sine and cosine functions are continuous and infinitely differentiable.

Theorem 10.3 (Properties of Sine and Cosine).

(a) *For any real number x we have $\cos(x)^2 + \sin(x)^2 = 1$. In particular, $\sin(x) \in [-1, 1]$ and $\cos(x) \in [-1, 1]$ for all real numbers x.*
(b) *For any real number x, we have $\sin'(x) = \cos(x)$, and $\cos'(x) = -\sin(x)$.*
(c) *For any real number x, we have $\sin(-x) = -\sin(x)$ and $\cos(-x) = \cos(x)$.*
(d) *For any real numbers x, y we have $\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y)$ and $\sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y)$.*
(e) *$\sin(0) = 0$ and $\cos(0) = 1$.*
(f) *For every real number x, we have $e^{ix} = \cos(x) + i\sin(x)$ and $e^{-ix} = \cos(x) - i\sin(x)$.*

Exercise 10.4. Prove Theorem 10.3. (Hints: whenever possible, write everything in terms of exponentials.)

Lemma 10.5. *There exists a positive real number x such that $\sin(x) = 0$.*

Proof. We argue by contradiction. Suppose $\sin(x) \neq 0$ for all $x > 0$. We conclude that $\cos(x) \neq 0$ for all $x > 0$, since $\cos(x) = 0$ implies that $\sin(2x) = 0$, by Theorem 10.3(d). Since $\cos(0) = 1$, we conclude that $\cos(x) > 0$ for all $x > 0$ by the Intermediate Value Theorem. Since $\sin(0) = 0$ and $\sin'(0) = 1 > 0$, we know that sin is positive for small positive x. Therefore, $\sin(x) > 0$ for all $x > 0$ by the Intermediate Value Theorem.

Define $\cot(x) := \cos(x)/\sin(x)$. Then cot is positive on $(0, \infty)$, and cot is differentiable for $x > 0$. From the quotient rule and Theorem 10.3(a), we have $\cot'(x) = -1/\sin^2(x)$. So, $\cot'(x) \leq -1$ for all $x > 0$. Then, by the Fundamental Theorem of Calculus, for all $x, s > 0$, we have $\cot(x + s) \leq \cot(x) - s$. Letting $s \to \infty$ shows that cot eventually becomes negative on $(0, \infty)$, a contradiction. □

Let E be the set $E := \{x \in (0,\infty)\colon \sin(x) = 0\}$, so that E is the set of zeros of the sine function. By Lemma 10.5, E is nonempty. Also, since sin is continuous, E is a closed set. (Note that $E = \sin^{-1}(0)$.) In particular, E contains all of its adherent points, so E contains $\inf(E)$.

Definition 10.6. We define π to be the number

$$\pi := \inf\{x \in (0,\infty)\colon \sin(x) = 0\}.$$

Then $\pi > 0$ and $\sin(\pi) = 0$. Since sin is nonzero on $(0,\pi)$ and $\sin'(0) = 1 > 0$, we conclude that sin is positive on $(0,\pi)$. Since $\cos'(x) = -\sin(x)$, we see that cos is decreasing on $(0,\pi)$. Since $\cos(0) = 1$, we therefore have $\cos(\pi) < 1$. Since $\sin^2(\pi) + \cos^2(\pi) = 1$ and $\sin(\pi) = 0$, we conclude that $\cos(\pi) = -1$.

We therefore deduce Euler's famous formula

$$e^{i\pi} = \cos(\pi) + i\sin(\pi) = -1.$$

Here are some more properties of sine and cosine.

Theorem 10.7.

(a) *For any real x we have $\cos(x+\pi) = -\cos(x)$ and $\sin(x+\pi) = -\sin(x)$. In particular, we have $\cos(x+2\pi) = \cos(x)$ and $\sin(x+2\pi) = \sin(x)$, so that* sin *and* cos *are 2π-periodic.*

(b) *If x is real, then $\sin(x) = 0$ if and only if x/π is an integer.*

(c) *If x is real, then $\cos(x) = 0$ if and only if x/π is an integer plus $1/2$.*

Exercise 10.8. Prove Theorem 10.7.

11. Appendix: Notation

Let A, B be sets in a space X. Let m, n be a nonnegative integers.

$$\begin{aligned}
\mathbb{Z} &:= \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}, \text{ the integers} \\
\mathbb{N} &:= \{0, 1, 2, 3, 4, 5, \ldots\}, \text{ the natural numbers} \\
\mathbb{Z}_+ &:= \{1, 2, 3, 4, \ldots\}, \text{ the positive integers} \\
\mathbb{Q} &:= \{m/n \colon m, n \in \mathbb{Z}, n \neq 0\}, \text{ the rationals} \\
\mathbb{R} &\text{ denotes the set of real numbers} \\
\mathbb{R}^* &= \mathbb{R} \cup \{-\infty\} \cup \{+\infty\} \text{ denotes the set of extended real numbers} \\
\mathbb{C} &:= \{x + y\sqrt{-1} : x, y \in \mathbb{R}\}, \text{ the complex numbers} \\
\emptyset &\text{ denotes the empty set, the set consisting of zero elements} \\
\in &\text{ means ``is an element of.'' For example, } 2 \in \mathbb{Z} \text{ is read as ``2 is an element of } \mathbb{Z}.\text{''} \\
\forall &\text{ means ``for all''} \\
\exists &\text{ means ``there exists''} \\
\mathbb{R}^n &:= \{(x_1, \ldots, x_n) \colon x_i \in \mathbb{R}, \forall\, i \in \{1, \ldots, n\}\} \\
A \subseteq B &\text{ means } \forall\; a \in A, \text{ we have } a \in B, \text{ so } A \text{ is contained in } B \\
A \smallsetminus B &:= \{x \in A \colon x \notin B\} \\
A^c &:= X \smallsetminus A, \text{ the complement of } A \\
A \cap B &\text{ denotes the intersection of } A \text{ and } B \\
A \cup B &\text{ denotes the union of } A \text{ and } B
\end{aligned}$$

Let (X, d) be a metric space, let $x_0 \in X$, let $r > 0$ be a real number, and let E be a subset of X. Let $(x_1, \ldots, x_n)$ be an element of $\mathbb{R}^n$, and let $p \geq 1$ be a real number.

$$\begin{aligned}
B_{(X,d)}(x_0, r) = B(x_0, r) &:= \{x \in X \colon d(x, x_0) < r\}. \\
\overline{E} &\text{ denotes the closure of } E \\
\mathrm{int}(E) &\text{ denotes the interior of } E \\
\partial E &\text{ denotes the boundary of } E
\end{aligned}$$

$$\begin{aligned}
\|(x_1, \ldots, x_n)\|_{\ell_p} &:= (\sum_{i=1}^{n} |x_i|^p)^{1/p} \\
\|(x_1, \ldots, x_n)\|_{\ell_\infty} &:= \max_{i=1,\ldots,n} |x_i|
\end{aligned}$$

Let $f, g\colon (X, d_X) \to (Y, d_Y)$ be maps between metric spaces. Let $V \subseteq X$, and let $W \subseteq Y$.

$$f(V) := \{f(v) \in Y \colon v \in V\}.$$

$$f^{-1}(W) := \{x \in X \colon f(x) \in W\}.$$

$$d_\infty(f, g) := \sup_{x \in X} d_Y(f(x), g(x)).$$

$B(X;Y)$ denotes the set of functions $f\colon X \to Y$ that are bounded.

$$C(X;Y) := \{f \in B(X;Y) \colon f \text{ is continuous}\}.$$

Let X be a set, and let $f\colon X \to \mathbb{C}$ be a complex-valued function.

$$\|f\|_\infty := \sup_{x \in X} |f(x)|.$$

11.1. **Set Theory.** Let X, Y be sets, and let $f\colon X \to Y$ be a function. The function $f\colon X \to Y$ is said to be **injective** (or **one-to-one**) if and only if: for every $x, x' \in V$, if $f(x) = f(x')$, then $x = x'$.

The function $f\colon X \to Y$ is said to be **surjective** (or **onto**) if and only if: for every $y \in Y$, there exists $x \in X$ such that $f(x) = y$.

The function $f\colon X \to Y$ is said to be **bijective** (or a **one-to-one correspondence**) if and only if: for every $y \in Y$, there exists exactly one $x \in X$ such that $f(x) = y$. A function $f\colon X \to Y$ is bijective if and only if it is both injective and surjective.

Two sets X, Y are said to have the same **cardinality** if and only if there exists a bijection from X onto Y.

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