3: REAL FUNCTIONS, CONTINUITY, DIFFERENTIABILITY

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Contents

1. Review

Proposition 1.1. *Let $(a_n)_{n=0}^{\infty}$ be a sequence of real numbers, and let L, L' be a real numbers with $L \neq L'$. Then $(a_n)_{n=0}^{\infty}$ cannot simultaneously converge to L and converge to L'.*

Theorem 1.2 (**Limit Laws**)**.** *Let $(a_n)_{n=0}^{\infty}$, $(b_n)_{n=0}^{\infty}$ be convergent sequences. Let x, y be real numbers such that $x = \lim_{n\to\infty} a_n$, $y = \lim_{n\to\infty} b_n$.*

(i) *The sequence $(a_n + b_n)$ converges to $x + y$. That is,*
$$\lim_{n\to\infty}(a_n + b_n) = (\lim_{n\to\infty} a_n) + (\lim_{n\to\infty} b_n).$$
(ii) *The sequence $(a_n b_n)$ converges to xy. That is,*
$$\lim_{n\to\infty}(a_n b_n) = (\lim_{n\to\infty} a_n)(\lim_{n\to\infty} b_n).$$
(iii) *For any real number c, the sequence (ca_n) converges to cx. That is,*
$$c\lim_{n\to\infty} a_n = \lim_{n\to\infty}(ca_n).$$
(iv) *The sequence $(a_n - b_n)$ converges to $x - y$. That is,*
$$\lim_{n\to\infty}(a_n - b_n) = (\lim_{n\to\infty} a_n) - (\lim_{n\to\infty} b_n).$$
(v) *Suppose $x \neq 0$ and there exists m such that $a_n \neq 0$ for all $n \geq m$. Then $(a_n^{-1})_{n=m}^{\infty}$ converges to x^{-1}. That is,*
$$\lim_{n\to\infty} a_n^{-1} = (\lim_{n\to\infty} a_n)^{-1}.$$
(vi) *Suppose $x \neq 0$ and there exists m such that $a_n \neq 0$ for all $n \geq m$. Then $(b_n/a_n)_{n=m}^{\infty}$ converges to y/x. That is,*
$$\lim_{n\to\infty}(b_n/a_n) = (\lim_{n\to\infty} b_n)/(\lim_{n\to\infty} a_n).$$
(vii) *Suppose $a_n \geq b_n$ for all $n \geq 0$. Then $x \geq y$.*

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Exercise 1.3. Let $(a_n)_{n=m}^{\infty}$ be a sequence of real numbers converging to 0. Show that $(|a_n|)_{n=m}^{\infty}$ also converges to zero.

Theorem 1.4 (**Bolzano-Weierstrass**). *Let $(a_n)_{n=0}^{\infty}$ be a bounded sequence. That is, there exists a real number M such that $|a_n| \leq M$ for all $n \in \mathbb{N}$. Then there exists a subsequence of $(a_n)_{n=0}^{\infty}$ which converges.*

Theorem 1.5 (**Least Upper Bound Property**). *Let E be a nonempty subset of $\mathbb{R}$. If E has some upper bound, then E has exactly one least upper bound.*

Lemma 1.6 (**Comparison Principle**). *Let $(a_n)_{n=m}^{\infty}$, $(b_n)_{n=m}^{\infty}$ be sequences of real numbers. Assume that $a_n \leq b_n$ for all $n \geq m$. Then*

- $\limsup_{n\to\infty} a_n \leq \limsup_{n\to\infty} b_n$.
- $\liminf_{n\to\infty} a_n \leq \liminf_{n\to\infty} b_n$.

In particular, $\lim_{n\to\infty} a_n$ *and* $\lim_{n\to\infty} b_n$ *exist, then* $\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n$.

Corollary 1.7 (**Squeeze Test/ Squeeze Theorem**). *Let $(a_n)_{n=m}^{\infty}$, $(b_n)_{n=m}^{\infty}$, $(c_n)_{n=m}^{\infty}$ be sequences of real numbers such that there exists a natural number M such that, for all $n \geq M$,*

$$a_n \leq b_n \leq c_n.$$

Assume that $(a_n)_{n=m}^{\infty}$ and $(c_n)_{n=m}^{\infty}$ converge to the same limit L. Then $(b_n)_{n=m}^{\infty}$ converges to L.

2. Functions on the real line

We now focus our attention on functions on the real line $\mathbb{R}$, rather than functions on $\mathbb{N}$ (i.e. sequences). The properties of the real line $\mathbb{R}$, most notably its completeness property, allow functions on $\mathbb{R}$ to have additional properties that functions on $\mathbb{N}$ do not have. For example, we can define and understand continuity and differentiability.

Definition 2.1. Let X, Y be sets and let $f : X \to Y$ be a **function**. That is, for every $x \in X$, the function f assigns to x some element $f(x) \in Y$. We say that X is the **domain** of f.

Example 2.2. Some common domains for functions on the real line are:

- The positive half-line $\mathbb{R}^+ := \{x \in \mathbb{R} : x > 0\}$.
- The negative half-line $\mathbb{R}^- := \{x \in \mathbb{R} : x < 0\}$.
- The **closed intervals** $[a, b] := \{x \in \mathbb{R} : a \leq x \leq b\}$, $a, b \in \mathbb{R}$.
- The **open intervals** $(a, b) := \{x \in \mathbb{R} : a < x < b\}$, $a, b \in \mathbb{R}$.
- The **half-open intervals** $(a, b] := \{x \in \mathbb{R} : a < x \leq b\}$ and $[a, b) := \{x \in \mathbb{R} : a \leq x < b\}$, $a, b \in \mathbb{R}$.
- $[a, \infty) := \{x \in \mathbb{R} : a \leq x < \infty\}$, $(-\infty, a] := \{x \in \mathbb{R} : -\infty < x \leq a\}$.
- $(a, \infty) := \{x \in \mathbb{R} : a < x < \infty\}$, $(-\infty, a) := \{x \in \mathbb{R} : -\infty < x < a\}$.
- The entire real line $\mathbb{R} = (-\infty, \infty)$.

Definition 2.3 (**Restriction**). Given a function $f : \mathbb{R} \to \mathbb{R}$ and given a subset $X \subseteq \mathbb{R}$, define the **restriction** $f|_X$ of f to X so that, for any $x \in X$, $f|_X(x) := f(x)$.

Remark 2.4. One can similarly restrict the range of a function, if the function only takes values in a smaller range. For example, the function $f(x) := x^2$ is a function $f : \mathbb{R} \to \mathbb{R}$, but it can also be considered as a function $f : \mathbb{R} \to [0, \infty)$.

Remark 2.5. There is a distinction between a function $f\colon \mathbb{R} \to \mathbb{R}$ and its value $f(x)$ for $x \in \mathbb{R}$, but it is not that important. For example, if we use $f(x) := x^2$ with $f\colon \mathbb{R} \to \mathbb{R}$, and we let $g := f|_{[0,1]}$, then $g(x) = f(x)$ for all $x \in [0,1]$. But f and g are not considered to be the same function, since their domains are different.

Definition 2.6 (Composition). Let $f\colon X \to Y$ and let $g\colon Y \to Z$ be functions. We define the **composition** $g \circ f$ by the formula $(g \circ f)(x) := g(f(x))$.

Definition 2.7 (Arithmetic of Functions). Real valued functions inherit the arithmetic of the real numbers as follows. Let $f, g\colon X \to \mathbb{R}$. Then the sum $(f+g)\colon X \to \mathbb{R}$ is defined so that, for all $x \in X$,

$$(f+g)(x) := f(x) + g(x).$$

The difference $(f-g)\colon X \to \mathbb{R}$ is defined so that, for all $x \in X$,

$$(f-g)(x) := f(x) - g(x).$$

The product $(fg)\colon X \to \mathbb{R}$ is defined so that, for all $x \in X$,

$$(fg)(x) := f(x)g(x).$$

If $g(x) \neq 0$ for all $x \in X$, then the quotient $(f/g)\colon X \to \mathbb{R}$ is defined so that, for all $x \in X$,

$$(f/g)(x) := f(x)/g(x).$$

If $c \in \mathbb{R}$, then the function $cf\colon X \to \mathbb{R}$ is defined so that, for all $x \in X$,

$$(cf)(x) := c(f(x)).$$

2.1. Limits of Functions.

Definition 2.8 (Adherent Point). Let E be a subset of $\mathbb{R}$, and let x be a real number. We say that x is an **adherent point** of E if and only if, for all $\varepsilon > 0$, there exists $y \in E$ such that $|x - y| < \varepsilon$.

Remark 2.9. All points in E are adherent points of E.

Definition 2.10 (Closure). Let E be a subset of $\mathbb{R}$. Then the **closure of** E, denoted $\overline{E}$, is defined to be the set of adherent points of E.

Proposition 2.11. *Let $a < b$ be real numbers. Let I be any of the four intervals (a,b), $(a,b]$, $[a,b)$ or $[a,b]$. Then the closure of I is $[a,b]$.*

Exercise 2.12. Prove Proposition 2.11.

Lemma 2.13. *Let X be a subset of $\mathbb{R}$, and let x be an element of $\mathbb{R}$. Then x is an adherent point of X if and only if there exists a sequence $(a_n)_{n=0}^{\infty}$ of elements of X such that $\lim_{n\to\infty} a_n = x$.*

Definition 2.14 (Convergence of a function). Let X be a subset of $\mathbb{R}$, let $f\colon X \to \mathbb{R}$ be a function, let E be a subset of X, let x_0 be an adherent point of E, and let L be a real number. We say that f **converges** to L at x_0 in E, and we write $\lim_{x \to x_0; x \in E} f(x) = L$ if and only if: for all $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon)$ such that, for all $x \in E$ with $|x - x_0| < \delta$, we have $|f(x) - L| < \varepsilon$.

If f does not converge to any real number L at x_0, we say that f **diverges** at x_0, and we leave $\lim_{x \to x_0; x \in E} f(x)$ undefined.

Remark 2.15. We will often omit the set E from our notation and just write $\lim_{x\to x_0} f(x)$. However, we must be careful when doing this.

We can equivalently talk about convergence of f in terms of sequences in the domain of f, as we now show.

Proposition 2.16. *Let X be a subset of $\mathbb{R}$, let $f\colon X \to \mathbb{R}$ be a function, let E be a subset of X, let x_0 be an adherent point of E, and let L be a real number. Then the following two statements are equivalent. (That is, one statement is true if and only if the other statement is true.)*

- *f converges to L at x_0 in E.*
- *For every sequence $(a_n)_{n=0}^{\infty}$ which consists entirely of elements of E, and which converges to x_0, the sequence $(f(a_n))_{n=0}^{\infty}$ converges to L.*

Exercise 2.17. Prove Proposition 2.16.

Remark 2.18. Due to Proposition 2.16, we will sometimes say "$f(x)$ goes to L as $x \to x_0$ in E" or "f has limit L at x_0 in E" instead of "f converges to L at x_0" or "$\lim_{x\to x_0} f(x) = L$".

Corollary 2.19. *Let X be a subset of $\mathbb{R}$, let $f\colon X \to \mathbb{R}$ be a function, let E be a subset of X, let x_0 be an adherent point of E. Then f can have at most one limit at x_0 in E.*

Proof. Suppose f has two limits L, L' at x_0 in E. We will show that $L = L'$. Since x_0 is an adherent point of E, Lemma 2.13 says that there exists a sequence $(a_n)_{n=0}^{\infty}$ of elements of E such that $a_n \to x_0$ as $n \to \infty$. By Proposition 2.16, the sequence $(f(a_n))_{n=0}^{\infty}$ converges to both L and L' as $n \to \infty$. By Proposition 1.1, we conclude that $L = L'$, as desired. □

By Proposition 2.16, the Limit Laws for sequences (Theorem 1.2) then give analogous limit laws for functions.

Proposition 2.20 (Limit Laws for functions). *Let X be a subset of $\mathbb{R}$, let $f, g\colon X \to \mathbb{R}$ be functions, let E be a subset of X, let x_0 be an adherent point of E, and let c be a real number. Assume that f has limit L at x_0 in E, and g has limit M at x_0 in E. Then $f+g$ has limit $L+M$ at x_0 in E, $f-g$ has limit $L-M$ at x_0 in E, fg has limit LM at x_0 in E, and cf has limit cL at x_0 in E. If additionally $g(x) \neq 0$ for all $x \in E$ and $M \neq 0$, then f/g has limit L/M at x_0 in E.*

Proof. We only prove the first claim, since the others are proven similarly. Since x_0 is an adherent point of E, Lemma 2.13 says that there exists a sequence $(a_n)_{n=0}^{\infty}$ of elements of E such that $a_n \to x_0$ as $n \to \infty$. By Proposition 2.16, the sequence $(f(a_n))_{n=0}^{\infty}$ converges to L. Similarly, the sequence $(g(a_n))_{n=0}^{\infty}$ converges to M. By the Limit Laws for sequences (Theorem 1.2), the sequence $(f(a_n)+g(a_n))_{n=0}^{\infty}$ converges to $L+M$. By Proposition 2.16, we conclude that $f+g$ has limit $L+M$ at x_0 in E. □

Remark 2.21. Let $c \in \mathbb{R}$. Using Proposition 2.16, we can verify the following limits

$$\lim_{x\to x_0; x\in\mathbb{R}} c = c.$$

$$\lim_{x\to x_0; x\in\mathbb{R}} x = x_0.$$

Then, using the limit laws of Proposition 2.16, we can e.g. compute

$$\lim_{x\to x_0; x\in\mathbb{R}} x^2 = x_0^2.$$

$$\lim_{x \to x_0; x \in \mathbb{R}} (x^2 + x) = x_0^2 + x_0.$$

Example 2.22. Let $f\colon \mathbb{R} \to \mathbb{R}$ so that

$$f(x) = \begin{cases} 1 & , \text{if } x > 0 \\ 0 & , \text{if } x \leq 0 \end{cases}.$$

Then $\lim_{x \to 0; x \in (0,\infty)} f(x) = 1$ and $\lim_{x \to 0; x \in (-\infty,0)} f(x) = 0$. However, $\lim_{x \to 0; x \in [0,\infty)} f(x)$ and $\lim_{x \to 0; x \in \mathbb{R}} f(x)$ are both undefined.

Example 2.23. Let $f\colon \mathbb{R} \to \mathbb{R}$ so that

$$f(x) = \begin{cases} 1 & , \text{if } x = 0 \\ 0 & , \text{if } x \neq 0 \end{cases}.$$

Then $\lim_{x \to 0; x \in \mathbb{R} \setminus \{0\}} f(x) = 0$, but $\lim_{x \to 0; x \in \mathbb{R}} f(x)$ is undefined.

Example 2.24. Let $f\colon \mathbb{R} \to \mathbb{R}$ so that

$$f(x) = \begin{cases} 1 & , \text{if } x \in \mathbb{Q} \\ 0 & , \text{if } x \notin \mathbb{Q} \end{cases}.$$

Then $\lim_{x \to 0; x \in \mathbb{R}} f(x)$ does not exist. To see this, consider the sequences $(1/n)_{n=1}^{\infty}$ and $(\sqrt{2}/n)_{n=1}^{\infty}$. Both sequences converge to zero as $n \to \infty$, though the first sequence consists of rational numbers, and the second sequence consists of irrational numbers. So, $f(1/n) \to 1$ as $n \to \infty$, while $f(\sqrt{2}/n) \to 0$ as $n \to \infty$. Therefore, $\lim_{x \to 0; x \in \mathbb{R}} f(x)$ does not exist.

The following proposition says that the limit of f at x_0 depends only on points near x_0.

Proposition 2.25. *Let X be a subset of $\mathbb{R}$, let $f\colon X \to \mathbb{R}$ be a function, let E be a subset of X, let x_0 be an adherent point of E, let L be a real number, and let δ be a positive real number. Then the following two statements are equivalent:*

- $\lim_{x \to x_0; x \in E} f(x) = L$.
- $\lim_{x \to x_0; x \in E \cap (x_0 - \delta, x_0 + \delta)} f(x) = L$.

Exercise 2.26. Prove Proposition 2.25.

3. Continuous Functions

As we saw from the examples in the previous section, there are many functions that behave very strangely with respect to limits. However, there are still large classes of functions that behave well with respect to limits. Such functions are called continuous.

When learning a new concept (such as continuous functions), it is often beneficial to consider various examples which satisfy or do not satisfy the properties of the new concept. We will therefore continue our family of examples from the previous section.

Definition 3.1 (**Continuous Function**)**.** Let X be a subset of $\mathbb{R}$ and let $f\colon X \to \mathbb{R}$ be a function. Let x_0 be an element of X. We say that f is **continuous** at x_0 if and only if

$$\lim_{x \to x_0; x \in X} f(x) = f(x_0).$$

That is, the limit of f at x_0 in X exists, and this limit is equal to $f(x_0)$. We say that f is **continuous on** X (or we just say that f is **continuous**) if and only if f is continuous at x_0

for every $x_0 \in X$. We say that f is **discontinuous** at x_0 if and only if f is not continuous at x_0.

Example 3.2. Let $f\colon \mathbb{R} \to \mathbb{R}$ so that

$$f(x) = \begin{cases} 1 & \text{, if } x > 0 \\ 0 & \text{, if } x \leq 0 \end{cases}.$$

Then f is continuous on $\mathbb{R} \setminus \{0\}$, but f is discontinuous at 0.

Example 3.3. Let $f\colon \mathbb{R} \to \mathbb{R}$ so that

$$f(x) = \begin{cases} 1 & \text{, if } x = 0 \\ 0 & \text{, if } x \neq 0 \end{cases}.$$

Then f is continuous on $\mathbb{R} \setminus \{0\}$, but f is discontinuous at 0. However, if we redefine f so that $f(0) := 0$, then f would be continuous on $\mathbb{R}$. We therefore say that f has a removable discontinuity at 0.

Example 3.4. Let $f\colon \mathbb{R} \to \mathbb{R}$ so that

$$f(x) = \begin{cases} 1 & \text{, if } x \in \mathbb{Q} \\ 0 & \text{, if } x \notin \mathbb{Q} \end{cases}.$$

As we saw previously, f is discontinuous at zero. In fact, f is discontinuous on all of $\mathbb{R}$.

Proposition 3.5. *Let X be a subset of $\mathbb{R}$, let $f\colon X \to \mathbb{R}$ be a function, and let $x_0 \in X$. Then the following three statements are equivalent.*

- *f is continuous at x_0*
- *For every sequence $(a_n)_{n=0}^{\infty}$ consisting of elements of X such that $\lim_{n\to\infty} a_n = x_0$, we have $\lim_{n\to\infty} f(a_n) = f(x_0)$.*
- *For every $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that, for all $x \in X$ with $|x - x_0| < \delta$, we have $|f(x) - f(x_0)| < \varepsilon$.*

Exercise 3.6. Prove Proposition 3.5

Proposition 3.7. *Let X be a subset of $\mathbb{R}$, and let $f, g\colon X \to \mathbb{R}$ be functions. Let $x_0 \in X$. If f, g are both continuous at x_0, then $f + g$ and $f \cdot g$ are continuous at x_0. If g is nonzero on X, then f/g is continuous at x_0.*

Proof. Apply the Limit Laws (Proposition 2.20) and Definition 3.1. □

Remark 3.8. Let $x, c \in \mathbb{R}$. Note that the constant function $f(x) := c$ and the function $f(x) := x$ are continuous. Then, Proposition 3.7 implies that polynomials are continuous, and rational functions are continuous whenever the denominator is nonzero. For example, the function $(x^2 + 1)/(x - 1)$ is continuous on $\mathbb{R} \setminus \{1\}$.

Proposition 3.9. *The function $f(x) := |x|$ is continuous on $\mathbb{R}$.*

Proof. Let $x_0 \in \mathbb{R}$. We split into three cases: $x_0 > 0$, $x_0 < 0$ and $x_0 = 0$. Suppose first that $x_0 > 0$. Define $\delta := |x_0|/2$. We show that f is continuous at x_0. From Proposition 2.25, it suffices to show that

$$x_0 = \lim_{x \to x_0; x \in (x_0 - \delta, x_0 + \delta)} f(x) = \lim_{x \to x_0; x \in (x_0/2, 3x_0/2)} f(x).$$

If $x \in (x_0/2, 3x_0/2)$, since $x_0 > 0$, we know that $x > 0$. So, $f(x) = x$. Therefore,

$$\lim_{x \to x_0; x \in (x_0/2, 3x_0/2)} f(x) = \lim_{x \to x_0; x \in (x_0/2, 3x_0/2)} x = x_0,$$

as desired. The case $x_0 < 0$ is similar.

We now conclude with the case $x_0 = 0$. Let $(a_n)_{n=0}^{\infty}$ be a sequence of real numbers converging to zero. From Proposition 3.5, it suffices to show that $(f(a_n))_{n=0}^{\infty}$ converges to zero. That is, it suffices to show: if $(a_n)_{n=0}^{\infty}$ converges to zero, then $(|a_n|)_{n=0}^{\infty}$ converges to zero. This follows from Exercise 1.3. □

Proposition 3.10. *Let X, Y be subsets of $\mathbb{R}$. Let $f: X \to Y$ and let $g: Y \to \mathbb{R}$ be functions. Let $x_0 \in X$. If f is continuous at x_0, and if g is continuous at $f(x_0)$, then $g \circ f$ is continuous at x_0.*

Exercise 3.11. Prove Proposition 3.10.

3.1. **Left and Right Limits.**

Definition 3.12. Let X be a subset of $\mathbb{R}$, let $f: X \to \mathbb{R}$ be a function, and let x_0 be a real number. If x_0 is an adherent point of $X \cap (x_0, \infty)$, then we define the **right limit** $f(x_0^+)$ of f at x_0 by the formula

$$f(x_0^+) := \lim_{x \to x_0; x \in X \cap (x_0, \infty)} f(x).$$

If this limit does not exist, or if x_0 is not an adherent point of $X \cap (x_0, \infty)$, we leave this limit undefined. Similarly, if x_0 is an adherent point of $X \cap (-\infty, x_0)$, then we define the **left limit** $f(x_0^-)$ of f at x_0 by the formula

$$f(x_0^-) := \lim_{x \to x_0; x \in X \cap (-\infty, x_0)} f(x).$$

If this limit does not exist, or if x_0 is not an adherent point of $X \cap (x_0, \infty)$, we leave this limit undefined.

Remark 3.13. Sometimes, we write $\lim_{x \to x_0^+} f(x)$ instead of $\lim_{x \to x_0; x \in X \cap (x_0, \infty)} f(x)$, and sometimes, we write $\lim_{x \to x_0^-} f(x)$ instead of $\lim_{x \to x_0; x \in X \cap (-\infty, x_0)} f(x)$.

The following proposition shows that, if both the left and right limits of a function exist at a point x_0, and if these limits are equal to $f(x_0)$, then f is continuous at x_0.

Proposition 3.14. *Let X be a subset of $\mathbb{R}$ containing a real number x_0. Suppose x_0 is an adherent point of both $X \cap (x_0, \infty)$ and $X \cap (-\infty, x_0)$. Let $f: X \to \mathbb{R}$ be a function. If $f(x_0^+)$ and $f(x_0^-)$ both exist, and we have $f(x_0^+) = f(x_0^-) = f(x_0)$, then f is continuous at x_0.*

3.2. **The Maximum Principle.**

We can now begin to prove some of properties of continuous functions. The Maximum Principle says that a continuous function on a closed interval $[a, b]$ achieves its maximum and minimum values on $[a, b]$.

Definition 3.15. Let X be a subset of $\mathbb{R}$, and let $f: X \to \mathbb{R}$ be a function. We say that f is **bounded from above** if and only if there exists a real number M such that $f(x) \leq M$ for all $x \in X$. We say that f is **bounded from below** if and only if there exists a real number M such that $f(x) \geq M$ for all $x \in X$. We say that f is **bounded** if and only if there exists a real number M such that $|f(x)| \leq M$ for all $x \in X$.

Remark 3.16. A function is bounded if and only if it is bounded from above and from below.

Remark 3.17. Some continuous functions are not bounded. For example, the function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) := x$ is unbounded on $\mathbb{R}$. Also, the function $f(x) := 1/x$ is unbounded on $(0,1)$.

However, if f is continuous on a closed interval, then it is automatically bounded, as we now show, using the Bolzano-Weierstrass Theorem in an indirect manner.

Lemma 3.18. *Let $a < b$ be real numbers. Let $f: [a,b] \to \mathbb{R}$ be a continuous function. Then f is bounded.*

Proof. We argue by contradiction. Assume f is not bounded. Then, for every natural number n, there exists a point $x_n \in [a,b]$ such that $|f(x_n)| > n$. Since the sequence $(x_n)_{n=0}^{\infty}$ is contained in the closed interval $[a,b]$, the Bolzano-Weierstrass Theorem (Theorem 1.4) shows that there exists a subsequence $(x_{n_j})_{j=0}^{\infty}$ of $(x_n)_{n=0}^{\infty}$ such that $(x_{n_j})_{j=0}^{\infty}$ converges to some real number y as $j \to \infty$. Note that $n_j \geq j$ by the definition of a subsequence. Since $(x_{n_j})_{j=0}^{\infty}$ is a convergent sequence contained in $[a,b]$, we know that y is an adherent point of $[a,b]$. From Proposition 2.11, we conclude that y is also in $[a,b]$, so that y is in the domain of f. Now, since f is continuous on $[a,b]$, it is continuous at y so

$$\lim_{j \to \infty} f(x_{n_j}) = f(y). \qquad (*)$$

Since $n_j \geq j$, the definition of the sequence $(x_n)_{n=0}^{\infty}$ shows that $\left|f(x_{n_j})\right| \geq n_j \geq j$. That is, for all natural numbers $j > 1+|f(y)|$, we have $\left|f(x_{n_j})\right| \geq j > 1+|f(y)|$. So, $\lim_{j\to\infty} f(x_{n_j}) \neq f(y)$, contradicting $(*)$. Since we have achieved a contradiction, the proof is concluded. ☐

Definition 3.19. Let $f: X \to \mathbb{R}$ be a function, and let $x_0 \in X$. We say that f **attains its maximum** at x_0 if and only if $f(x_0) \geq f(x)$ for all $x \in X$. We say that f **attains its minimum** at x_0 if and only if $f(x_0) \leq f(x)$ for all $x \in X$.

We can now modify the proof of Lemma 3.18 a bit to give a stronger statement.

Theorem 3.20 (The Maximum Principle). *Let $a < b$ be real numbers and let $f: [a,b] \to \mathbb{R}$ be a function that is continuous on $[a,b]$. Then f attains its maximum and minimum on $[a,b]$.*

Proof. We will show that f attains its maximum on $[a,b]$. Such a result applied to $-f$ then implies that f also attains its minimum on $[a,b]$.

From Lemma 3.18, there exists a real number M such that $-M \leq f(x) \leq M$ for all $x \in [a,b]$. Define

$$E := f([a,b]) = \{f(x) \colon x \in [a,b]\}.$$

Note that E is a nonempty subset of $\mathbb{R}$ that is bounded from above (and below). From the Least Upper Bound property (Theorem 1.5), E has a least upper bound $S := \sup(E)$.

For each positive integer n, the real number $S - 1/n$ is not an upper bound for E, since S is the least upper bound of E. So, there exists some $x_n \in [a,b]$ such that $f(x_n) \geq S - 1/n$. We are now once again in a position to apply the Bolzano-Weierstrass Theorem. Since the sequence $(x_n)_{n=1}^{\infty}$ is contained in the closed interval $[a,b]$, the Bolzano-Weierstrass Theorem (Theorem 1.4) shows that there exists a subsequence $(x_{n_j})_{j=1}^{\infty}$ of $(x_n)_{n=1}^{\infty}$ such that $(x_{n_j})_{j=1}^{\infty}$

converges to some real number y as $j \to \infty$. Note that $n_j \geq j$ by the definition of a subsequence, so $-1/n_j \geq -1/j$. Since $(x_{n_j})_{j=1}^{\infty}$ is a convergent sequence contained in $[a,b]$, we know that y is an adherent point of $[a,b]$. From Proposition 2.11, we conclude that y is also in $[a,b]$, so that y is in the domain of f. Now, since f is continuous on $[a,b]$, it is continuous at y so

$$\lim_{j\to\infty} f(x_{n_j}) = f(y). \qquad (*)$$

Since $n_j \geq j$, the definition of the sequence $(x_n)_{n=1}^{\infty}$ shows that

$$f(x_{n_j}) \geq S - 1/n_j \geq S - 1/j.$$

Also, since S is the supremum of f, we have $f(x_{n_j}) \leq S$. So, letting $j \to \infty$ and using the Squeeze Theorem (Corollary 1.7), we conclude that $S = \lim_{j\to\infty} f(x_{n_j}) = f(y)$, as desired. □

Remark 3.21. For a function $f\colon [a,b] \to \mathbb{R}$, we write $\sup_{x\in[a,b]} f(x)$ as shorthand for $\sup\{f(x)\colon x \in [a,b]\}$, and we write $\inf_{x\in[a,b]} f(x)$ as shorthand for $\inf\{f(x)\colon x \in [a,b]\}$

Remark 3.22. The assumptions of Theorem 3.20 cannot be weakened in general. For example, consider the function $f(x) := x$ on the open interval $(0,1)$. Then $\sup_{x\in(0,1)} f(x) = 1$ and $\inf_{x\in(0,1)} f(x) = 0$, but f does not take the value 1 or 0 on the open interval $(0,1)$, even though f is continuous.

Also, consider the function $f\colon [-1,1] \to \mathbb{R}$ defined by

$$f(x) := \begin{cases} x+1 & \text{, if } x \in [-1,0) \\ 0 & \text{, if } x = 0 \\ x-1 & \text{, if } x \in (0,1] \end{cases}.$$

Then $\sup_{x\in[-1,1]} f(x) = 1$ and $\inf_{x\in[-1,1]} f(x) = -1$, but f does not take the value 1 or -1 on the closed interval $[-1,1]$. Note that f is discontinuous at $x = 0$, so Theorem 3.20 does not apply.

3.3. **The Intermediate Value Theorem.** From Theorem 3.20, we know that a continuous function $f\colon [a,b] \to \mathbb{R}$ attains its minimum and maximum on $[a,b]$. We now show that f also attains all values in between the maximum and minimum.

Theorem 3.23 (Intermediate Value Theorem). *Let $a < b$ be real numbers. Let $f\colon [a,b] \to \mathbb{R}$ be function that is continuous on $[a,b]$. Let y be a real number between $f(a)$ and $f(b)$, so that either $f(a) \leq y \leq f(b)$ or $f(a) \geq y \geq f(b)$. Then there exists a $c \in [a,b]$ such that $f(c) = y$.*

Proof. Without loss of generality, assume that $f(a) \leq y \leq f(b)$. If $y = f(a)$ or $y = f(b)$, we just set $c = a$ or $c = b$ as needed. We therefore assume that $f(a) < y < f(b)$. Define

$$E := \{x \in [a,b]\colon f(x) < y\}.$$

Since $f(a) < y$, E is nonempty. Since E is contained in $[a,b]$, E is bounded from above. By the Least Upper Bound property (Theorem 1.5), E has a least upper bound $c := \sup(E)$. We will prove that $f(c) = y$.

Since b is an upper bound for E, we know that $c \leq b$. Since $a \in E$, we know that $a \leq c$. So, $c \in [a,b]$. By looking to the left of c, we will show that $f(c) \leq y$, and then by looking to the right of c, we will show that $f(c) \geq y$.

We now show that $f(c) \leq y$. Let n be a positive integer. Then $c - 1/n < c = \sup(E)$, so $c-1/n$ is not an upper bound for E. So, there exists a point $x_n \in E$ such that $x_n > c-1/n$. Since c is an upper bound for E, $x_n \leq c$. So

$$c - 1/n \leq x_n \leq c.$$

Letting $n \to \infty$, we conclude by the Squeeze Theorem (Corollary 1.7) that $\lim_{n\to\infty} x_n = c$. Since f is continuous at c, we have $\lim_{n\to\infty} f(x_n) = f(c)$. Since $x_n \in E$ for every positive integer n, we have $f(x_n) < y$ for every positive integer n. By the Comparison Principle (Lemma 1.6), we conclude that

$$f(c) = \lim_{n\to\infty} f(x_n) \leq y.$$

We now show that $f(c) \geq y$. Since $f(c) \leq y < f(b)$, we have $c \neq b$. Since $c \in [a, b]$, we then have $c < b$. So, there exists a positive integer m such that, for all $n \geq m$, $c + 1/n < b$. Then $c + 1/n > c$. Since $c = \sup(E)$, we conclude that $c + 1/n \notin E$. Also, $c + 1/n \in [a, b]$. So, by the definition of E, we have $f(c + 1/n) \geq y$. Since f is continuous at c, we have $\lim_{n\to\infty} f(c+1/n) = f(c)$. By the Comparison Principle (Lemma 1.6), we conclude that

$$f(c) = \lim_{n\to\infty} f(c+1/n) \geq y.$$

Finally, $y \leq f(c) \leq y$, so $f(c) = y$, as desired. □

Remark 3.24. The assumption that f is continuous is necessary for Theorem 3.23. For example, consider the function

$$f(x) := \begin{cases} 0 & \text{, if } x < 0 \\ 1 & \text{, if } x \geq 0 \end{cases}.$$

Remark 3.25. Theorem 3.23 gives another way to prove the existence of n^{th} roots. For example, for $x \in \mathbb{R}$, define $f(x) := x^2$, $f\colon [0, 2] \to \mathbb{R}$. Then $f(0) = 0$, $f(2) = 4$, so choosing $y = 2$, there exists at least one $c \in [0, 2]$ such that $f(c) = c^2 = 2$.

Corollary 3.26. *Let $a < b$ be real numbers. Let $f\colon [a, b] \to \mathbb{R}$ be a continuous function on $[a, b]$. Let $M := \sup_{x\in[a,b]} f(x)$ be the maximum value of f on $[a, b]$, and let $m := \inf_{x\in[a,b]} f(x)$ be the minimum value of f on $[a, b]$. Let y be a real number such that $m \leq y \leq M$. Then there exists $c \in [a, b]$ such that $f(c) = y$. Moreover, $f([a, b]) = [m, M]$.*

Exercise 3.27. Prove Corollary 3.26.

3.4. **Monotone Functions.**

Definition 3.28. Let X be a subset of $\mathbb{R}$ and let $f\colon X \to \mathbb{R}$ be a function. We say that f is **monotone increasing** if and only if $f(y) \geq f(x)$ for all $x, y \in X$ with $y > x$. We say that f is **strictly monotone increasing** if and only if $f(y) > f(x)$ for all $x, y \in X$ with $y > x$. Similarly, we say that f is **monotone decreasing** if and only if $f(y) \leq f(x)$ for all $x, y \in X$ with $y > x$. We say that f is **strictly monotone decreasing** if and only if $f(y) < f(x)$ for all $x, y \in X$ with $y > x$. We say that f is **monotone** if and only if it is either monotone increasing or monotone decreasing. We say that f is **strictly monotone** if and only if it is either strictly monotone increasing or strictly monotone decreasing.

A strictly monotone and continuous function has a continuous inverse, as we now show.

Proposition 3.29. *Let $a < b$ be real numbers, and let $f\colon [a, b] \to \mathbb{R}$ be a function which is both continuous and strictly monotone increasing. Then f is a bijection from $[a, b]$ to $[f(a), f(b)]$, and the inverse function $f^{-1}\colon [f(a), f(b)] \to [a, b]$ is also continuous and strictly monotone increasing.*

Exercise 3.30. Prove Proposition 3.29. (Hint: To prove that f^{-1} is continuous, use the ε–δ definition of continuity.)

3.5. **Uniform Continuity.** There is a bit of an odd point in the definition of continuity. A function $f\colon \mathbb{R} \to \mathbb{R}$ is continuous if and only if it is continuous at every $x \in \mathbb{R}$. That is, given any $x_0 \in \mathbb{R}$ and any $\varepsilon > 0$, there exists a $\delta = \delta(x_0, \varepsilon)$ such that, if $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \varepsilon$. Note in particular that δ may depend on x_0. For example, the function $f\colon (0, \infty) \to \mathbb{R}$ defined by $f(x) := 1/x$ is continuous on $(0, \infty)$, but f is not bounded. The problem here is that, if $\varepsilon > 0$ is fixed, then $\delta(x_0, \varepsilon)$ must be chosen to be smaller and smaller as $x_0 \to 0^+$. It would be nicer if we could select δ in a way that does not depend on x_0, as in the following definition.

Definition 3.31 (**Uniform Continuity**). Let X be a subset of $\mathbb{R}$, and let $f\colon X \to \mathbb{R}$ be a function. We say that f is **uniformly continuous** if and only if, for every $\varepsilon > 0$ there exists $\delta > 0$ such that, if $x, x_0 \in X$ satisfy $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \varepsilon$.

Remark 3.32. A uniformly continuous function is continuous.

Example 3.33. The function $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) := x$ is uniformly continuous. On the other hand, the function $f\colon (0, \infty) \to \mathbb{R}$ defined by $f(x) := 1/x$ is not uniformly continuous.

Just as in the case of continuity, there is a way to characterize uniform continuity using sequences. We now explore this characterization.

Definition 3.34. Let $(a_n)_{n=m}^{\infty}, (b_n)_{n=m}^{\infty}$ be two sequences of real numbers. We say that $(a_n)_{n=m}^{\infty}$ and $(b_n)_{n=m}^{\infty}$ are **equivalent** if and only if for every real $\varepsilon > 0$, there exists an integer $N = N(\varepsilon) > m$ such that, for all $n \geq N$, we have $|a_n - b_n| < \varepsilon$.

Lemma 3.35. *Let $(a_n)_{n=m}^{\infty}, (b_n)_{n=m}^{\infty}$ be two sequences of real numbers. Then $(a_n)_{n=m}^{\infty}$ and $(b_n)_{n=m}^{\infty}$ are equivalent if and only if $\lim_{n\to\infty}(a_n - b_n) = 0$.*

Exercise 3.36. Prove Lemma 3.35.

Note that equivalent sequences need not converge.

Proposition 3.37. *Let X be a subset of $\mathbb{R}$ and let $f\colon X \to \mathbb{R}$ be a function. Then the following two statements are equivalent.*

- *f is uniformly continuous on X.*
- *For any two equivalent sequences $(a_n)_{n=m}^{\infty}, (b_n)_{n=m}^{\infty}$, the sequences $(f(a_n))_{n=m}^{\infty}, (f(b_n))_{n=m}^{\infty}$ are also equivalent sequences.*

Exercise 3.38. Prove Proposition 3.37.

Remark 3.39. From Proposition 3.5, we saw that continuous functions map convergent sequences to convergent sequences. Proposition 3.37 then says that uniformly continuous functions map equivalent sequences to equivalent sequences.

Corollary 3.40. *Let X be a subset of $\mathbb{R}$ and let $f\colon X \to \mathbb{R}$ be a uniformly continuous function. Let x_0 be an adherent point of X. Then $\lim_{x\to x_0} f(x)$ exists (and so it is a real number.)*

Exercise 3.41. Prove Corollary 3.40

Remark 3.42. Note that Corollary 3.40 is false in general, if f is just continuous. For example, consider again $f(x) := 1/x$, where $f\colon (0,\infty) \to \mathbb{R}$. Then $\lim_{x\to 0^+} f(x)$ does not exist. But also recall that f is not uniformly continuous.

Uniformly continuous functions also map bounded sets to bounded sets.

Proposition 3.43. *Let X be a subset of $\mathbb{R}$, and let $f\colon X \to \mathbb{R}$ be a uniformly continuous function. Assume that E is a bounded subset of X. Then $f(E)$ is also bounded.*

Exercise 3.44. Prove Proposition 3.43.

Since uniformly continuous functions have such nice properties, it is helpful to have some conditions to easily verify uniform continuity, as in the following Theorem.

Theorem 3.45. *Let $a < b$ be real numbers, and let $f\colon [a,b] \to \mathbb{R}$ be a function which is continuous on $[a,b]$. Then f is also uniformly continuous on $[a,b]$.*

Proof. We argue by contradiction. Suppose f is not uniformly continuous on $[a,b]$. So, using Proposition 3.37, there exist two equivalent sequences $(a_n)_{n=m}^{\infty}, (b_n)_{n=m}^{\infty}$ contained in $[a,b]$ such that $(f(a_n))_{n=m}^{\infty}, (f(b_n))_{n=m}^{\infty}$ are not equivalent. That is, there exists an $\varepsilon > 0$ such that, for all integers $N > m$, there exists an integer $n \geq N$ such that

$$|f(a_n) - f(b_n)| \geq \varepsilon. \qquad (*)$$

In particular, the following set is infinite

$$A := \{n \in \mathbb{N} \colon\ |f(a_n) - f(b_n)| \geq \varepsilon\}.$$

That is, given any set of natural numbers $n_0 < n_1 < \cdots < n_j$ in A, there exists an integer $n_{j+1} > n_j$ so that $\left|f(a_{n_j}) - f(b_{n_j})\right| \geq \varepsilon$. So, consider the sequences $(a_{n_j})_{j=0}^{\infty}, (b_{n_j})_{j=0}^{\infty}$ which are equivalent and contained in $[a,b]$. By the Bolzano-Weierstrass Theorem, there exists a subsequence $(a_{n_{j_k}})_{k=0}^{\infty}$ of $(a_{n_j})_{j=0}^{\infty}$ such that $(a_{n_{j_k}})_{k=0}^{\infty}$ converges as $k \to \infty$. From Lemma 3.35, since $(a_{n_{j_k}})_{k=0}^{\infty}$ and $(b_{n_{j_k}})_{k=0}^{\infty}$ are equivalent sequences, we conclude that $(b_{n_{j_k}})_{k=0}^{\infty}$ converges as $k \to \infty$ as well. Using Lemma 3.35 again, $(a_{n_{j_k}})_{k=0}^{\infty}$ and $(b_{n_{j_k}})_{k=0}^{\infty}$ converge to the same point $c \in [a,b]$. So, using the Limit Laws (Proposition 2.20),

$$\lim_{k\to\infty} (f(a_{n_{j_k}}) - f(b_{n_{j_k}})) = 0$$

Since this violates $(*)$, we have achieved a contradiction, concluding the proof. □

3.6. **Limits at Infinity.**

Definition 3.46. Let X be a subset of $\mathbb{R}$. We say that $+\infty$ is an adherent point of X if and only if for every $M \in \mathbb{R}$ there exists an $x \in X$ such that $x > M$. We say that $-\infty$ is an adherent point of X if and only if for every $M \in \mathbb{R}$ there exists an $x \in X$ such that $x < M$.

Definition 3.47. Let X be a subset of $\mathbb{R}$ such that $+\infty$ is an adherent point of X. Let $f\colon X \to \mathbb{R}$ be a function and let L be a real number. We say that $f(x)$ **converges to** L as $x \to +\infty$ if and only if, for every $\varepsilon > 0$, there exists a real M such that, for all $x \in X$ with $x > M$, we have $|f(x) - L| < \varepsilon$. Similarly, if $-\infty$ is an adherent point of X, then we say that $f(x)$ **converges to** L as $x \to -\infty$ if and only if, for every $\varepsilon > 0$, there exists a real M such that, for all $x \in X$ with $x < M$, we have $|f(x) - L| < \varepsilon$.

Example 3.48. Let $f\colon (0, \infty) \to \mathbb{R}$ be defined by $f(x) := 1/x$. Then $\lim_{x\to+\infty} f(x) = 0$.

4. Derivatives

We will soon define a derivative, but before doing so, we adjust slightly the definition of adherent point.

Definition 4.1. Let X be a subset of $\mathbb{R}$ and let x be a real number. We say that x is a **limit point** of X (or x is a **cluster point** of X) if and only if x is an adherent point of $X \smallsetminus \{x\}$.

Remark 4.2. That is, x is a limit point of X if and only if, for every real $\varepsilon > 0$, there exists a $y \in X$ with $y \neq x$ such that $|y - x| < \varepsilon$.

Lemma 2.13 then implies the following.

Lemma 4.3. *Let X be a subset of $\mathbb{R}$, and let x be a real number. Then x is a limit point of X if and only if there exists a sequence $(a_n)_{n=m}^{\infty}$ of elements of $X \smallsetminus \{x\}$ such that $(a_n)_{n=m}^{\infty}$ converges to x.*

Lemma 4.4. *Let I be a (possibly infinite) interval. That is, I is equal to a set of the form (a, b), $[a, b]$, $(a, b]$, $[a, b)$, $(a, +\infty)$, $[a, +\infty)$, $(-\infty, b)$, $(-\infty, b]$ or $(-\infty, \infty)$ where $a, b \in \mathbb{R}$ and $a < b$. Then every element of I is a limit point of I.*

Proof. We only prove the case $I = [a, b]$ and leave the rest as exercises.

Suppose $x \in [a, b)$. Then there exists a positive integer N such that, for all $n \geq N$, $x + 1/n < b$. So, the sequence $(x + 1/n)_{n=N}^{\infty}$ is contained in $I \smallsetminus \{x\}$, and this sequence converges to x. Therefore, x is a limit point of $[a, b]$, by Lemma 4.3. To deal with the remaining case of $x = b$, we do the same thing but we use the sequence $(x - 1/n)_{n=N}^{\infty}$. □

We can now define derivatives.

Definition 4.5. Let X be a subset of $\mathbb{R}$, and let x_0 be an element of X which is also a limit point of X. Let $f\colon X \to \mathbb{R}$ be a function. If the limit

$$\lim_{x\to x_0; x\in X\smallsetminus\{x_0\}} \frac{f(x) - f(x_0)}{x - x_0}.$$

converges to a real number L, then we say that f is **differentiable** at x_0 on X **with derivative** L, and we write $f'(x_0) := L$. If this limit does not exist, or if x_0 is not a limit point of X, we leave $f'(x_0)$ undefined, and we say that f is **not differentiable** at x_0 on X.

Remark 4.6. Note that we need x_0 to be a limit point of $X \smallsetminus \{x_0\}$, otherwise the limit in the definition of the derivative would be undefined. Often, the set X will be an interval as in Lemma 4.4, so this issue will not arise.

Example 4.7. Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by $f(x) := x$. Then

$$f'(x_0) = \lim_{x \to x_0; x \in \mathbb{R} \smallsetminus \{x_0\}} \frac{x - x_0}{x - x_0} = 1.$$

Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by $f(x) := x^2$. Then

$$\begin{aligned} f'(x_0) &= \lim_{x \to x_0; x \in \mathbb{R} \smallsetminus \{x_0\}} \frac{x^2 - x_0^2}{x - x_0} = \lim_{x \to x_0; x \in \mathbb{R} \smallsetminus \{x_0\}} \frac{(x + x_0)(x - x_0)}{x - x_0} \\ &= \lim_{x \to x_0; x \in \mathbb{R} \smallsetminus \{x_0\}} (x + x_0) = 2x_0. \end{aligned}$$

In general, if k is a positive integer, and if $f(x) := x^k$, $f\colon \mathbb{R} \to \mathbb{R}$, then

$$\begin{aligned} f'(x_0) &= \lim_{x \to x_0; x \in \mathbb{R} \smallsetminus \{x_0\}} \frac{x^k - x_0^k}{x - x_0} = \lim_{x \to x_0; x \in \mathbb{R} \smallsetminus \{x_0\}} \frac{(\sum_{j=1}^k x^{k-j} x_0^{j-1})(x - x_0)}{x - x_0} \\ &= \lim_{x \to x_0; x \in \mathbb{R} \smallsetminus \{x_0\}} \sum_{j=1}^k x^{k-j} x_0^{j-1} = \sum_{j=1}^k x_0^{k-1} = k x_0^{k-1}. \end{aligned}$$

Remark 4.8. Sometimes one writes $f'(x)$ as df/dx, but we will not do so here.

We now give an example of a continuous function that is not differentiable at zero.

Example 4.9. Define $f(x) := |x|$. For $x_0 \in (-\infty, 0) \cup (0, \infty)$, one can show that f is differentiable. However, f is not differentiable at 0. To see this, observe that

$$\lim_{x \to 0; x \in (0,\infty)} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0; x \in (0,\infty)} \frac{x - f(0)}{x - 0} = 1.$$

$$\lim_{x \to 0; x \in (-\infty,0)} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0; x \in (-\infty,0)} \frac{-x - f(0)}{x - 0} = -1.$$

Therefore, $\lim_{x \to 0; x \in \mathbb{R} \smallsetminus \{0\}} \frac{f(x) - f(0)}{x - 0}$ does not exist. So, f is not differentiable at 0.

Even though a function may be continuous but not differentiable at a point, a function that is differentiable at a point is always continuous at that point.

Proposition 4.10. *Let X be a subset of $\mathbb{R}$, let x_0 be a limit point of X, and let $f\colon X \to \mathbb{R}$ be a function. If f is differentiable at x_0, then f is also continuous at x_0.*

Exercise 4.11. Prove Proposition 4.10

If a function is differentiable at x_0, then it is approximately linear at x_0 in the following sense.

Proposition 4.12. *Let X be a subset of $\mathbb{R}$, let x_0 be a limit point of X, let $f\colon X \to \mathbb{R}$ be a function, and let L be a real number. Then the following two statements are equivalent.*

- *f is differentiable at x_0 on X with derivative L.*
- *For every $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that, if $x \in X$ satisfies $|x - x_0| < \delta$, then*

$$|f(x) - [f(x_0) + L(x - x_0)]| \leq \varepsilon |x - x_0|.$$

Exercise 4.13. Prove Proposition 4.12.

Remark 4.14. The second item is understood informally as $f(x) \approx f(x_0) + f'(x_0)(x - x_0)$.

Definition 4.15. Let X be a subset of $\mathbb{R}$ and let $f: X \to \mathbb{R}$ be a function. We say that f is **differentiable** on X if and only if f is differentiable at x_0 for all $x_0 \in X$.

Using this definition and Proposition 4.10, we get the following.

Corollary 4.16. *Let X be a subset of $\mathbb{R}$ and let $f: X \to \mathbb{R}$ be a function that is differentiable on X. Then f is continuous on X.*

Theorem 4.17 (**Properties of Derivatives**). *Let X be a subset of $\mathbb{R}$, let x_0 be a limit point of X, and let $f: X \to \mathbb{R}$ and $g: X \to \mathbb{R}$ be functions.*

(i) *If f is constant, so that there exists $c \in \mathbb{R}$ such that $f(x) = c$ for all $x \in X$, then f is differentiable at x_0 and $f'(x_0) = 0$.*
(ii) *If f is the identity function, so that $f(x) = x$ for al $x \in X$, then f is differentiable at x_0 and $f'(x_0) = 1$.*
(iii) *If f, g are differentiable at x_0, then $f + g$ is differentiable at x_0, and $(f+g)'(x_0) = f'(x_0) + g'(x_0)$. (**Sum Rule**)*
(iv) *If f, g are differentiable at x_0, then fg is differentiable at x_0, and $(fg)'(x_0) = f'(x_0)g(x_0) + g'(x_0)f(x_0)$. (**Product Rule**)*
(v) *If f is differentiable at x_0, and if $c \in \mathbb{R}$, then cf is differentiable at x_0, and $(cf)'(x_0) = cf'(x_0)$.*
(vi) *If f, g are differentiable at x_0, then $f - g$ is differentiable at x_0, and $(f-g)'(x_0) = f'(x_0) - g'(x_0)$.*
(vii) *If g is differentiable at x_0, and if $g(x) \neq 0$ for all $x \in X$, then $1/g$ is differentiable at x_0, and $(1/g)'(x_0) = -\frac{g'(x_0)}{(g(x_0))^2}$.*
(viii) *If f, g are differentiable at x_0, and if $g(x) \neq 0$ for all $x \in X$, then f/g is differentiable at x_0, and*

$$(f/g)'(x_0) = \frac{g(x_0)f'(x_0) - f(x_0)g'(x_0)}{(g(x_0))^2}. \qquad (\textbf{\textit{Quotient Rule}})$$

Exercise 4.18. Prove Theorem 4.17. For the product rule, you may need the following identity

$$f(x)g(x) - f(x_0)g(x_0) = f(x)(g(x) - g(x_0)) + g(x_0)(f(x) - f(x_0)).$$

Theorem 4.19 (**Chain Rule**). *Let X, Y be subsets of $\mathbb{R}$, let $x_0 \in X$ be a limit point of X, and let $y_0 \in Y$ be a limit point of Y. Let $f: X \to Y$ be a function such that $f(x_0) = y_0$ and such that f is differentiable at x_0. Let $g: Y \to \mathbb{R}$ be a function that is differentiable at y_0. Then the function $g \circ f: X \to \mathbb{R}$ is differentiable at x_0, and*

$$(g \circ f)'(x_0) = g'(y_0)f'(x_0).$$

Exercise 4.20. Prove Theorem 4.19. (Hint: using Proposition 2.16, it suffices to consider a sequence $(a_n)_{n=0}^{\infty}$ of elements of X converging to x_0. Also, from Proposition 4.10, f is continuous, so $(f(a_n))_{n=0}^{\infty}$ converges to $f(x_0)$.)

4.1. **Local Extrema.**

Definition 4.21. Let $f: X \to \mathbb{R}$ be a function, and let $x_0 \in X$. We say that f **attains a local maximum** at x_0 if and only if there exists a $\delta > 0$ such that the restriction $f|_{X \cap (x_0 - \delta, x_0 + \delta)}$ attains a maximum at x_0. We say that f **attains a local minimum** at x_0 if and only if there exists a $\delta > 0$ such that the restriction $f|_{X \cap (x_0 - \delta, x_0 + \delta)}$ attains a minimum at x_0.

Remark 4.22. If $f\colon X \to \mathbb{R}$ attains a maximum at x_0, then we sometimes say that f attains a **global maximum** at x_0.

Proposition 4.23. *Let $a < b$ be real numbers, and let $f\colon (a,b) \to \mathbb{R}$ be a function. If $x_0 \in (a,b)$, if f is differentiable at x_0, and if f attains a local maximum or minimum at x_0, then $f'(x_0) = 0$.*

Exercise 4.24. Prove Proposition 4.23.

Remark 4.25. Note that Proposition 4.23 is not true if f we assume that $f\colon [a,b] \to \mathbb{R}$ achieves a local maximum or minimum. For example, the function $f\colon [0,1] \to \mathbb{R}$ defined by $f(x) := x$ satisfies $f'(x) = 1$ for all $x \in [0,1]$, while f achieves a local maximum at $x = 1$ and a local minimum at $x = 0$.

Theorem 4.26 (Rolle's Theorem). *Let $a < b$ be real numbers, and let $f\colon [a,b] \to \mathbb{R}$ be a continuous function which is differentiable on (a,b). Assume that $f(a) = f(b)$. Then there exists $x \in (a,b)$ such that $f'(x) = 0$.*

Exercise 4.27. Prove Theorem 4.26. (Hint: use Proposition 4.23 and the Maximum Principle, Theorem 3.20.)

Theorem 4.26 then has the following useful corollary.

Corollary 4.28 (Mean Value Theorem). *Let $a < b$ be real numbers, and let $f\colon [a,b] \to \mathbb{R}$ be a continuous function which is differentiable on (a,b). Then there exists $x \in (a,b)$ such that*

$$f'(x) = \frac{f(b) - f(a)}{b - a}.$$

Proof. Consider the function $g\colon [a,b] \to \mathbb{R}$ defined by

$$g(y) := f(y) - \frac{f(b) - f(a)}{b - a}(y - a). \qquad (*)$$

Note that $g(a) = f(a) = g(b)$, g is continuous on $[a,b]$ by Proposition 3.7, and g is differentiable on (a,b) by Theorem 4.17(v) and (iii). So by Theorem 4.26, there exists $x \in (a,b)$ such that $g'(x) = 0$. Using $(*)$ and Theorem 4.17, $g'(x) = 0$ says that

$$0 = f'(x) - \frac{f(b) - f(a)}{b - a}.$$

□

4.2. **Monotone Functions and Derivatives.** We now explore the connection between the monotonicity of a function and the sign of its derivative.

Proposition 4.29. *Let X be a subset of $\mathbb{R}$, let x_0 be a limit point of X, and let $f\colon X \to \mathbb{R}$ be a function. If f is monotone increasing and if f is differentiable at x_0, then $f'(x_0) \geq 0$. If f is monotone decreasing and if f is differentiable at x_0, then $f'(x_0) \leq 0$.*

Exercise 4.30. Prove Proposition 4.29.

Remark 4.31. Note that we need to assume that f is both monotone and differentiable, since there exist functions that are monotone but not differentiable. Consider for example $f\colon \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) := \begin{cases} 0 & \text{, if } x < 0 \\ 1 & \text{, if } x \geq 0 \end{cases}.$$

A strictly monotone increasing function can have a zero derivative. Consider for example $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) := x^3$, and note that $f'(0) = 0$. However, a converse statement is true, as we now show.

Proposition 4.32. *Let $a < b$ be real numbers, and let $f\colon [a,b] \to \mathbb{R}$ be a differentiable function. If $f'(x) > 0$ for all $x \in [a,b]$, then f is strictly monotone increasing. If $f'(x) < 0$ for all $x \in [a,b]$, then f is strictly monotone decreasing. If $f'(x) = 0$ for all $x \in [a,b]$, then f is a constant function.*

Exercise 4.33. Prove Proposition 4.32. (Hint: for the final statement, use the Mean-Value Theorem.)

4.3. **Inverse Functions and Derivatives.** Let X, Y be subsets of $\mathbb{R}$. If we have a bijective function $f\colon X \to Y$ which is differentiable, then the derivative of f^{-1} is related nicely to the derivative of f, as we now show.

Lemma 4.34. *Let X, Y be subsets of $\mathbb{R}$. Let $f\colon X \to Y$ be a bijection, so that $f^{-1}\colon Y \to X$ is a function. Let $x_0 \in X$ and $y_0 \in Y$ such that $f(x_0) = y_0$. (Consequently, $x_0 = f^{-1}(y_0)$.) If f is differentiable at x_0 and if f^{-1} is differentiable at y_0, then $f'(x_0) \neq 0$ and*

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)}.$$

Proof. Note that $(f^{-1} \circ f)(x) = x$ for all $x \in X$. So, from the Theorem 4.17(ii) and the Chain Rule (Theorem 4.19),

$$1 = (f^{-1} \circ f)'(x_0) = (f^{-1})'(y_0) f'(x_0).$$

Since $(f^{-1})'(y_0) f'(x_0) = 1$, we know that $f'(x_0) \neq 0$, and $(f^{-1})'(y_0) = 1/f'(x_0)$ □

Remark 4.35. As a consequence of Lemma 4.34, we see that if f is differentiable at x_0 with $f'(x_0) = 0$, then f^{-1} is not differentiable at $y_0 = f(x_0)$. For example, consider the function $f(x) := x^n$, where n is a positive integer and $f\colon [0,\infty) \to [0,\infty)$. Then $f^{-1}(x) = x^{1/n}$, $f^{-1}\colon [0,\infty) \to [0,\infty)$. And if $n \geq 2$, then $f'(0) = 0$, so f^{-1} is not differentiable at 0.

Lemma 4.34 is deficient, in that we need to assume that f^{-1} is differentiable at $f(x_0)$. It would be more preferable to know that f^{-1} is differentiable by only using information about f. Such a goal is accomplished in the following theorem.

Theorem 4.36 (**Inverse Function Theorem**)**.** *Let X, Y be subsets of $\mathbb{R}$. Let $f\colon X \to Y$ be bijection, so that $f^{-1}\colon Y \to X$ is a function. Let $x_0 \in X$ and $y_0 \in Y$ such that $f(x_0) = y_0$. If f is differentiable at x_0, if f^{-1} is continuous at y_0, and if $f'(x_0) \neq 0$, then f^{-1} is differentiable at y_0 with*

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)}.$$

Proof. We are required to show that

$$\lim_{y\to y_0; y\in Y\setminus\{y_0\}} \frac{f^{-1}(y)-f^{-1}(y_0)}{y-y_0} = \frac{1}{f'(x_0)}.$$

By Proposition 2.16, given any sequence $(y_n)_{n=1}^{\infty}$ of elements in $Y \setminus \{y_0\}$ that converges to y_0, it suffices to show that

$$\lim_{n\to\infty} \frac{f^{-1}(y_n)-f^{-1}(y_0)}{y_n-y_0} = \frac{1}{f'(x_0)}. \qquad (*)$$

Note that f is a bijection, so there exists a sequence of elements $(x_n)_{n=1}^{\infty}$ such that $f(x_n) = y_n$ for all $n \geq 1$. Moreover, since $(y_n)_{n=1}^{\infty}$ is contained in $Y \setminus \{y_0\}$, since $f(x_0) = y_0$, and since f is a bijection, the sequence $(x_n)_{n=1}^{\infty}$ is contained in $X \setminus \{x_0\}$. So, since f is differentiable at x_0, we have by Proposition 2.16 that

$$\lim_{n\to\infty} \frac{f(x_n)-f(x_0)}{x_n-x_0} = f'(x_0).$$

That is,

$$\lim_{n\to\infty} \frac{y_n-y_0}{f^{-1}(y_n)-f^{-1}(y_0)} = f'(x_0). \qquad (**)$$

Since $y_n \neq y_0$ for all $n \geq 1$, the numerator on the left of $(**)$ is nonzero. Also, by hypothesis, $f'(x_0) \neq 0$. So, we can invert both sides of $(**)$ and apply the limit laws (Theorem 1.2(v)) to conclude that $(*)$ holds, as desired. $\square$

5. Appendix: Notation

Let A, B be sets in a space X. Let m, n be a nonnegative integers.

$\mathbb{Z} := \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$, the integers
$\mathbb{N} := \{0, 1, 2, 3, 4, 5, \ldots\}$, the natural numbers
$\mathbb{Z}_+ := \{1, 2, 3, 4, \ldots\}$, the positive integers
$\mathbb{Q} := \{m/n \colon m, n \in \mathbb{Z}, n \neq 0\}$, the rationals
$\mathbb{R}$ denotes the set of real numbers
$\mathbb{R}^* = \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$ denotes the set of extended real numbers
$\mathbb{C} := \{x + y\sqrt{-1} : x, y \in \mathbb{R}\}$, the complex numbers
$\emptyset$ denotes the empty set, the set consisting of zero elements
$\in$ means "is an element of." For example, $2 \in \mathbb{Z}$ is read as "2 is an element of $\mathbb{Z}$."

$$\forall \text{ means "for all"}$$
$$\exists \text{ means "there exists"}$$
$$\mathbb{F}^n := \{(x_1, \ldots, x_n) \colon x_i \in \mathbb{F}, \forall\, i \in \{1, \ldots, n\}\}$$
$$A \subseteq B \text{ means } \forall\ a \in A, \text{ we have } a \in B, \text{ so } A \text{ is contained in } B$$
$$A \smallsetminus B := \{x \in A \colon x \notin B\}$$
$$A^c := X \smallsetminus A, \text{ the complement of } A$$
$$A \cap B \text{ denotes the intersection of } A \text{ and } B$$
$$A \cup B \text{ denotes the union of } A \text{ and } B$$

Let E be a subset of $\mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$. Let $(a_n)_{n=0}^{\infty}$ be a sequence of real numbers.

$$\sup(E) \text{ denotes the smallest upper bound of } E$$
$$\inf(E) \text{ denotes the largest lower bound of } E$$
$$\limsup (a_n)_{n=0}^{\infty} := \lim_{n\to\infty} \sup_{m\geq n} (a_n)_{n=m}^{\infty}$$
$$\liminf (a_n)_{n=0}^{\infty} := \lim_{n\to\infty} \inf_{m\geq n} (a_n)_{n=m}^{\infty}$$

5.1. **Set Theory.** Let X, Y be sets, and let $f \colon X \to Y$ be a function. The function $f \colon X \to Y$ is said to be **injective** (or **one-to-one**) if and only if: for every $x, x' \in V$, if $f(x) = f(x')$, then $x = x'$.

The function $f \colon X \to Y$ is said to be **surjective** (or **onto**) if and only if: for every $y \in Y$, there exists $x \in X$ such that $f(x) = y$.

The function $f \colon X \to Y$ is said to be **bijective** (or a **one-to-one correspondence**) if and only if: for every $y \in Y$, there exists exactly one $x \in X$ such that $f(x) = y$. A function $f \colon X \to Y$ is bijective if and only if it is both injective and surjective.

Two sets X, Y are said to have the same **cardinality** if and only if there exists a bijection from X onto Y.

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