Please provide complete and well-written solutions to the following exercises.

Due March 29, at the beginning of class.

Homework 8

Exercise 1. Let $k \geq 1$ be an integer. Let $X: \Omega \to \mathbf{R}$ be a random variable with finite k^{th} moment: $\mathbf{E}|X|^k < \infty$. Show that $\phi_X(t)$ is k-times continuously differentiable in t, and

$$\frac{d^k}{dt^k}|_{t=0}\phi_X(t) = i^k \mathbf{E}X^k.$$

In particular, we get the Taylor expansion

$$\phi_X(t) = \sum_{n=0}^{k} \frac{(it)^n}{n!}\mathbf{E}X^n + o(|t|^k), \qquad \forall\, t \in \mathbf{R}.$$

Let $f, g: \mathbf{R} \to \mathbf{R}$. We use the notation $f(t) = o(g(t))$, $\forall\, t \in \mathbf{R}$ to denote $\lim_{t\to 0}\left|\frac{f(t)}{g(t)}\right| = 0$.

Exercise 2. Assume that $X: \Omega \to \mathbf{R}$ is a **subgaussian** random variable, i.e. $\exists\, a, b > 0$ such that

$$\mathbf{P}(|X| > t) \leq a e^{-bt^2}, \qquad \forall\, t \in \mathbf{R}.$$

Show that ϕ_X is equal to its Taylor series:

$$\phi_X(t) = \sum_{n=0}^{\infty} \frac{(it)^n}{n!}\mathbf{E}X^n, \qquad \forall\, t \in \mathbf{R}.$$

Also, show that the Taylor series converges uniformly on any closed interval.

Exercise 3. Let $X, Y: \Omega \to \mathbf{R}$ be independent random variables. Show that

$$\phi_{X+Y}(t) = \phi_X(t)\phi_Y(t), \qquad \forall\, t \in \mathbf{R}.$$

Exercise 4 (Lévy Continuity Theorem)**.** Let $X, X_1, X_2, \ldots$ be real-valued random variables (possibly on different sample spaces). Assume that, $\forall\, t \in \mathbf{R}$, $\phi(t) := \lim_{n\to\infty} \phi_{X_n}(t)$ exists. Then the following are equivalent.

- (i) ϕ is continuous at 0.
- (ii) $\mu_{X_1}, \mu_{X_2}, \ldots$ is tight. ($\forall\, \varepsilon > 0$, $\exists\, m = m(\varepsilon) > 0$ such that $\limsup_{n\to\infty}(1 - \mu_{X_n}([-m, m])) \leq \varepsilon$.)
- (iii) There exists a random variable X such that $\phi_X = \phi$.
- (iv) $X_1, X_2, \ldots$ converges in distribution to X.

(Hint: Use the Exercise from the previous homework that characterizes tightness for vague convergence to get from (ii) to other conditions.)

Exercise 5. Let $f, g, h\colon \mathbf{R} \to \mathbf{C}$. We use the notation $f(s) = o(g(s))\ \forall\ s \in \mathbf{R}$ to denote $\lim_{s\to 0} \left|\frac{f(s)}{g(s)}\right| = 0$. For example, if $f(s) = s^3\ \forall\ s \in \mathbf{R}$, then $f(s) = o(s^2)$, since $\lim_{s\to 0} \left|\frac{f(s)}{s^2}\right| = \lim_{s\to 0} |s| = 0$. Show: (i) if $f(s) = o(g(s))$ and if $h(s) = o(g(s))$, then $(f + h)(s) = o(g(s))$. (ii) If c is any nonzero constant, then $o(cg(s)) = o(g(s))$. (iii) $\lim_{s\to 0} g(s)o(1/g(s)) = 0$. (iv) $\lim_{s\to 0} o(g(s))/g(s) = 0$. (v) $o(g(s) + o(g(s))) = o(g(s))$.

Exercise 6. Give an alternate proof of the fact $\mathcal{F}[e^{-x^2/2}](\xi) = \sqrt{2\pi}e^{-\xi^2/2}$ using the following strategy:

- Let $g(\xi) := (2\pi)^{-1/2}\mathcal{F}[e^{-x^2/2}](\xi)$. Show that $g'(\xi) = -\xi g(\xi)$ for all $\xi \in \mathbf{R}$.
- Deduce that $(d/d\xi)(g(\xi)e^{\xi^2/2}) = 0$.
- Finally, conclude that $g(\xi) = e^{-\xi^2/2}$.

Exercise 7 (Weak Berry-Esséen theorem)**.** Let $X, X_1, X_2, \ldots$ be i.i.d. real-valued random variables with mean zero, variance 1 and with $\mathbf{E}\,|X|^3 < \infty$. Let Z be a standard Gaussian random variable.

(i) Show that for any compactly supported $g\colon \mathbf{R} \to \mathbf{R}$ with three continuous derivatives, and for any $n \geq 1$,

$$\mathbf{E}g\Big(\frac{X_1 + \cdots + X_n}{\sqrt{n}}\Big) = \mathbf{E}g(Z) + O(n^{-1/2} \sup_{x\in\mathbf{R}} |g'''(x)|\, \mathbf{E}|X|^3),$$

where the implied constant does not depend on g, n or on any of the random variables $X, X_1, X_2, \ldots$.

(ii) Show that, for any $n \geq 1$, and for any $t \in \mathbf{R}$

$$\mathbf{P}\Big(\frac{X_1 + \cdots + X_n}{\sqrt{n}} \leq t\Big) = \mathbf{P}(Z \leq t) + O(n^{-1/2}\mathbf{E}|X|^3)^{1/4},$$

where the implied constant does not depend on n, t or on any of the random variables $X, X_1, X_2, \ldots$.

(Hint: for the first item, try to modify the argument we did in class.) (Hint: for the second item, consider $g = 1_{[-\infty,t]} * \phi_\varepsilon$, where $\phi(x) = e^{-x^2/2}/\sqrt{2\pi}$ and $\phi_\varepsilon(x) = \varepsilon^{-1}\phi(x/\varepsilon)$ for any $x \in \mathbf{R}$ and for appropriately chosen $\varepsilon > 0$.)

Exercise 8. Let $\phi\colon \mathbf{R} \to \mathbf{R}$ be a Schwartz function. Let Z be a standard Gaussian random variable. In applications of Stein's method, it is sometimes more convenient to take another derivative of Stein's identity, resulting in the following Ornstein-Uhlenbeck identities.

- $\mathbf{E}[\phi''(Z) - Z\phi'(Z)] = 0$.
- If $h\colon \mathbf{R} \to \mathbf{R}$ is a Schwartz function, then the function

  $$g(x) := \int_0^1 \frac{1}{2t}[\mathbf{E}h(x\sqrt{t} + Z\sqrt{1-t}) - \mathbf{E}h(Z)]dt, \qquad \forall\, x \in \mathbf{R},$$

  is a solution of the differential equation

  $$h(x) - \mathbf{E}h(Z) = g''(x) - xg'(x), \qquad \forall\, x \in \mathbf{R}.$$

Exercise 9. Using the Central Limit Theorem, prove the Weak Law of Large Numbers (assume the random variables have mean zero and variance one).

Exercise 10. Show that there exists a nonzero random variable X such that, if $X_1, X_2, \ldots$ are i.i.d. copies of X, then $\frac{X_1+\cdots+X_n}{n}$ is equal in distribution to X, for any $n \geq 1$. (Optional: can you write out an explicit formula for the density of X?) (Hint: take the Fourier transform.)

Show that there exists a nonzero random variable X such that, if $X_1, X_2, \ldots$ are i.i.d. copies of X, then $\frac{X_1+\cdots+X_n}{n^2}$ is equal in distribution to X, for any $n \geq 1$.

Exercise 11. Let $Z = (Z_1, \ldots, Z_d) \in \mathbf{R}^d$ be a Gaussian random vector.

- Show that the covariance matrix $(a_{ij})_{1\leq i,j\leq d}$ of Z is symmetric, positive semidefinite. That is, for any $v \in \mathbf{R}^d$, we have

$$v^T a v = \sum_{i,j=1}^{d} v_i v_j a_{ij} \geq 0.$$

- Given any symmetric positive semidefinite matrix $(b_{ij})_{1\leq i,j\leq d}$, show that there exists a Gaussian random vector Z such that the covariance matrix of Z is $(b_{ij})_{1\leq i,j\leq d}$. (Hint: write the matrix b in its Cholesky decomposition $b = rr^*$, where r is a $d \times d$ real matrix. Let $e^{(1)}, \ldots, e^{(d)}$ be the rows of r. Let $X_1, \ldots, X_d$ be independent standard Gaussian random variables. Let $X := (X_1, \ldots, X_d)$. Define $Z_i := \langle X, e^{(i)} \rangle$ for any $1 \leq i \leq d$.)