Please provide complete and well-written solutions to the following exercises.

Due March 22, at the beginning of class.

Homework 7

Exercise 1. Show that $\cosh(x) \leq e^{x^2/2}$, $\forall\, x \in \mathbf{R}$.

Exercise 2 (Chernoff Inequality)**.** Let $0 < p < 1$. Let $X_1, X_2, \ldots$ be independent identically distributed random variables with $\mathbf{P}(X_1 = 1) = p$ and $\mathbf{P}(X_1 = 0) = 1 - p$ for any $i \geq 1$. Then for any $n \geq 1$

$$\mathbf{P}\Big(\frac{1}{n}\sum_{i=1}^{n} X_i \geq t\Big) \leq e^{-np}\Big(\frac{ep}{t}\Big)^{tn}, \qquad \forall\, t \geq p.$$

Prove the same estimate for $\mathbf{P}(\frac{1}{n}\sum_{i=1}^{n} X_i \leq t)$ for any $t \leq p$. (Hint: $1 + x \leq e^x$ for any $x \in \mathbf{R}$, so $1 + (e^\alpha - 1)p \leq e^{(e^\alpha - 1)p}$.)

Exercise 3. We return to the Erdös-Renyi random graph $G = (V, E)$ on n vertices with parameter $0 < p < 1$ from an earlier homework. Define $d := p(n-1)$.

- Show that d is the expected degree of each vertex in G. (The degree of a vertex $v \in V$ is the number of vertices connected to v by an edge in E.)
- Show that there exists a constant $c > 0$ such that the following holds. Assume $p \geq \frac{c \log n}{n}$. Then with probability larger than .9, all vertices of G have degrees in the range $(.9d, 1.1d)$. (Hint: first consider a single vertex, then use the union bound over all vertices.)

Exercise 4 (Khintchine Inequality)**.** Let $0 < p < \infty$. Then there exist constants $A_p, B_p \in (0, \infty)$ such that the following holds.

Let $X_1, X_2, \ldots$ be independent identically distributed random variables with $\mathbf{P}(X_1 = 1) = \mathbf{P}(X_1 = -1) = 1/2$. Let $a_1, a_2, \ldots \in \mathbf{R}$. Then

$$A_p \left|\left|\sum_{i=1}^{n} a_i X_i\right|\right|_p \leq \left|\left|\sum_{i=1}^{n} a_i X_i\right|\right|_2 = (\sum_{i=1}^{n} a_i^2)^{1/2} \leq B_p \left|\left|\sum_{i=1}^{n} a_i X_i\right|\right|_p.$$

So, all L_p (quasi)-norms of $\sum_{i=1}^{n} a_i X_i$ are comparable.

(In Banach space terminology, there is an isomorphic copy of the Banach space ℓ_2 inside any space $L_p[0,1]$; e.g. we can use $X_i(t) := \operatorname{sign}\sin(2^i \pi t)$ for any $t \in [0,1]$, $i \geq 1$.)

(Hint: For the A_p inequality, use Hoeffding's inequality and "Integration by Parts," obtaining $A_p \leq \sqrt{p}A$ for some fixed $A > 0$. For the B_p inequality with $0 < p < 2$, apply Logarithmic Convexity of L_p norms, in the form $||X||_2^2 \leq ||X||_p^{2(1-\theta)}\, ||X||_4^{2\theta}$, then apply the A_4 inequality to get $||X||_2^{2(1-\theta)} \leq A_p\, ||X||_p^{2(1-\theta)}$.)

Exercise 5. Let $X_1, X_2, \ldots : \Omega \to \mathbf{R}$ be i.i.d. with $\mathbf{E}|X_1| = \infty$. Then $\mathbf{P}(|X_n| > n$ for infinitely many $n \geq 1) = 1$. And $\mathbf{P}(\lim_{n\to\infty} \frac{X_1+\cdots+X_n}{n} \in (-\infty, \infty)) = 0$. (Hint: show $\sum_{n=1}^{\infty} \mathbf{P}(|X_n| > n) = \infty$, then apply the second Borel-Cantelli Lemma. Write $\frac{S_n}{n} - \frac{S_{n+1}}{n+1} = \frac{S_n}{n(n+1)} - \frac{X_{n+1}}{n+1}$, and consider what happens to both sides on the set where $\lim_{n\to\infty} \frac{S_n}{n} \in \mathbf{R}$.)

Also, unfortunately the strong law cannot hold for triangular arrays.

Exercise 6. Let X be a random variable taking values in the natural numbers with $\mathbf{P}(X = n) = \frac{1}{\zeta(3)} \frac{1}{n^3}$, where $\zeta(3) := \sum_{m=1}^{\infty} \frac{1}{m^3}$.

- Show that X is absolutely integrable.
- For any $n \geq 1$, let $X_{n,1}, \ldots, X_{n,n} \colon \Omega \to \mathbf{R}$ be independent copies of X. Show that the random variables $\frac{X_{n,1}+\cdots+X_{n,n}}{n}$ are almost surely unbounded. (Hint: for any constant c, show that $\frac{X_{n,1}+\cdots+X_{n,n}}{n} > c$ occurs with probability at least ε/n for some $\varepsilon > 0$ depending on c. Then use the second Borel-Cantelli lemma.)

Exercise 7 (Second Borel-Cantelli Lemma)**.** Let $A_1, A_2, \ldots$ be independent events with $\sum_{n=1}^{\infty} \mathbf{P}(A_n) = \infty$. Then $\mathbf{P}(A_n$ occurs for infinitely many $n \geq 1) = 1$. (Hint: using $1 - x \leq e^{-x}$ for any $x \in \mathbf{R}$, show $\mathbf{P}(\cap_{n=s}^{t} A_n^c) \leq \exp(-\sum_{n=s}^{t} \mathbf{P}(A_n))$, let $t \to \infty$ to conclude $\mathbf{P}(\cup_{n=s}^{\infty} A_n) = 1$ for all $s \geq 1$, then let $s \to \infty$.)

Exercise 8. Let $X, X_1, X_2, \ldots$ and let $Y, Y_1, Y_2, \ldots$ be random variables with values in $\mathbf{R}$.

(i) Assume that X is constant almost surely. Show that $X_1, X_2, \ldots$ converges to X in distribution if and only if $X_1, X_2, \ldots$ converges to X in probability.

(ii) Prove this Lemma from the notes: Let $\mu_1, \mu_2, \ldots$ be a sequence of probability measures on $\mathbf{R}$. Then any subsequential limit of the sequence (with respect to vague convergence) is a probability measure if and only if $\mu_1, \mu_2, \ldots$ is **tight**: $\forall\ \varepsilon > 0$, $\exists\ m = m(\varepsilon) > 0$ such that

$$\limsup_{n\to\infty}(1 - \mu_n([-m, m])) \leq \varepsilon.$$

(iii) Suppose that $X_1, X_2, \ldots$ converges in distribution to X. Show there exist random variables $Z, Z_1, Z_2, \ldots : \Omega \to \mathbf{R}$ such that $\mu_Z = \mu_X$, $\mu_{Z_n} = \mu_{X_n}$ for any $n \geq 1$, and such that $Z_1, Z_2, \ldots$ converges almost surely to Z. (Hint: use the sample space $\Omega = [0, 1]$ and using an exercise from a previous homework, represent each random variable on Ω as the "inverse" of its cumulative distribution function.)

(iv) (Slutsky's Theorem) Suppose $X_1, X_2, \ldots$ converges in distribution to X and $Y_1, Y_2, \ldots$ converges in probability to Y. Assume Y is constant almost surely. Show that $X_1+Y_1, X_2+Y_2, \ldots$ converges in distribution to $X+Y$. Show also that $X_1Y_1, X_2Y_2, \ldots$ converges in distribution to XY. (Hint: either use (iii) or use (ii) to control error terms.) What happens if Y is not constant almost surely?

(v) (Fatou's lemma) If $g \colon \mathbf{R} \to [0, \infty)$ is continuous, and if $X_1, X_2, \ldots$ converges in distribution to X, show that $\liminf_{n\to\infty} \mathbf{E}g(X_n) \geq \mathbf{E}g(X)$.

(vi) (Bounded convergence) If $g \colon \mathbf{R} \to \mathbf{C}$ is continuous and bounded, and if $X_1, X_2, \ldots$ converges in distribution to X, show that $\lim_{n\to\infty} \mathbf{E}g(X_n) = \mathbf{E}g(X)$.

(vii) (Dominated convergence) If $X_1, X_2, \ldots\colon \Omega \to \mathbf{R}$ converges in distribution to X, and if there exists a random variable $Y\colon \Omega \to [0,\infty)$ with $|X_n| \leq Y$ for all $n \geq 1$ and $\mathbf{E}Y < \infty$, show that $\lim_{n\to\infty} \mathbf{E}X_n = \mathbf{E}X$.

Exercise 9 (Portmanteau Theorem)**.** Let $X, X_1, X_2, \ldots$ be random variables with values in $\mathbf{R}$. Show that the condition ($X_1, X_2, \ldots$ converges in distribution to X) is equivalent to the following three statements:

- For any closed $K \subseteq \mathbf{R}$, $\limsup_{n\to\infty} \mathbf{P}(X_n \in K) \leq \mathbf{P}(X \in K)$.
- For any open $U \subseteq \mathbf{R}$, $\liminf_{n\to\infty} \mathbf{P}(X_n \in U) \leq \mathbf{P}(X \in U)$.
- For any Borel set $E \subseteq \mathbf{R}$ whose topological boundary ∂E satisfies $\mathbf{P}(X \in \partial E) = 0$, $\lim_{n\to\infty} \mathbf{P}(X_n \in E) = \mathbf{P}(X \in E)$.

(Hint: Urysohn's Lemma might be helpful.)

Exercise 10. Let $f, g, h\colon \mathbf{R} \to \mathbf{R}$ be measurable functions. Assume that $\int_{\mathbf{R}} |f(x)|\, dx$, $\int_{\mathbf{R}} |g(x)|\, dx < \infty$ and $\int_{\mathbf{R}} |h(x)|\, dx < \infty$. Show that $\int_{-\infty}^{\infty} |(g * h)(t)|\, dt < \infty$. Consequently, $(g * h)(t) \in \mathbf{R}$ almost surely for $t \in \mathbf{R}$ (with respect to Lebesgue measure on $\mathbf{R}$).

Then, show that convolution is associative and commutative. That is, $g * h = h * g$ and $f * (g * h) = (f * g) * h$ almost surely.

Exercise 11. Using convolution, show that if X, Y are standard Gaussian random variables, then $aX + bY$ is a Gaussian random variable with mean 0 and variance $a^2 + b^2$.

Exercise 12. Let X, Y, Z be independent and uniformly distributed on $[0, 1]$. Note that f_X is not a continuous function.

Using convolution, compute f_{X+Y}. Draw f_{X+Y}. Note that f_{X+Y} is a continuous function, but it is not differentiable at some points.

Using convolution, compute f_{X+Y+Z}. Draw f_{X+Y+Z}. Note that f_{X+Y+Z} is a differentiable function, but it does not have a second derivative at some points.

Make a conjecture about how many derivatives $f_{X_1+\cdots+X_n}$ has, where $X_1, \ldots, X_n$ are independent and uniformly distributed on $[0, 1]$. You do not have to prove this conjecture. The idea of this exercise is that convolution is a kind of average of functions. And the more averaging you do, the more derivatives $f_{X_1+\cdots+X_n}$ has. Lastly, $f_{X_1+\cdots+X_n}$ should resemble a Gaussian density when n becomes large. So, we should be able to guess at a formulation of the Central Limit Theorem, at least for i.i.d. random variables with density.

Exercise 13. Construct two random variables X, Y such that X and Y are each uniformly distributed on $[0, 1]$, and such that $\mathbf{P}(X + Y = 1) = 1$.

Then construct two random variables W, Z such that W and Z are each uniformly distributed on $[0, 1]$, and such that $W + Z$ is uniformly distributed on $[0, 2]$.

(Hint: there is a way to do each of the above problems with about one line of work. That is, there is a way to solve each problem without working very hard.)