Please provide complete and well-written solutions to the following exercises.

Due February 22, at the beginning of class.

Homework 5

Exercise 1. Let $Y_1, Y_2, \ldots \colon \Omega \to \mathbf{R}$ be random variables that converge almost surely to a random variable $Y \colon \Omega \to \mathbf{R}$. Show that $Y_1, Y_2, \ldots$ converges in probability to Y in the following way.

- For any $\varepsilon > 0$ and for any positive integer n, let
$$A_{n,\varepsilon} := \bigcup_{m=n}^{\infty} \{\omega \in \Omega \colon\ |Y_m(\omega) - Y(\omega)| > \varepsilon\}.$$
Show that $A_{n,\varepsilon} \supseteq A_{n+1,\varepsilon} \supseteq A_{n+2,\varepsilon} \supseteq \cdots$.
- Show that $\mathbf{P}(\cap_{n=1}^{\infty} A_{n,\varepsilon}) = 0$.
- Using Continuity of the Probability Law, deduce that $\lim_{n\to\infty} \mathbf{P}(A_{n,\varepsilon}) = 0$.

Now, show that the converse is false. That is, find random variables $Y_1, Y_2, \ldots$ that converge in probability to Y, but where $Y_1, Y_2, \ldots$ do not converge to Y almost surely.

Exercise 2. Let $0 < p \leq \infty$. Show that, if $Y_1, Y_2, \ldots \colon \Omega \to \mathbf{R}$ converge to $Y \colon \Omega \to \mathbf{R}$ in L_p, then $Y_1, Y_2, \ldots$ converges to Y in probability.

Then, show that the converse is false.

Exercise 3. Suppose random variables $Y_1, Y_2, \ldots \colon \Omega \to \mathbf{R}$ converge in probability to a random variable $Y \colon \Omega \to \mathbf{R}$. Prove that $Y_1, Y_2, \ldots$ converge in distribution to Y.

Then, show that the converse is false.

Exercise 4. Prove the following statement. Almost sure convergence does not imply convergence in L_2, and convergence in L_2 does not imply almost sure convergence. That is, find random variables that converge in L_2 but not almost surely. Then, find random variables that converge almost surely but not in L_2.

Exercise 5. Let $X, X_1, X_2, \ldots \colon \Omega \to \mathbf{R}$.

(i) Suppose that $\sum_{i=1}^{\infty} \mathbf{P}(|X_i - X| > \varepsilon) < \infty$ for all $\varepsilon > 0$. Show that $X_1, X_2, \ldots$ converges to X almost surely. Show that the converse does not hold in general.

(ii) Suppose $X_1, X_2, \ldots$ converges to X in probability. Show there is a subsequence $X_{i_1}, X_{i_2}, \ldots$ of $X_1, X_2, \ldots$ such that $X_{i_1}, X_{i_2}, \ldots$ converges to X almost surely. (Here $i_1 < i_2 < \cdots$)

(iii) (Urysohn subsequence principle) Suppose that every subsequence $X_{i_1}, X_{i_2}, \ldots$ of $X_1, X_2, \ldots$ has a further subsequence $X_{i_{j_1}}, X_{i_{j_2}}, \ldots$ that converges to X in probability. Show that $X_1, X_2, \ldots$ also converges to X in probability.

(iv) Suppose $X_1, X_2, \ldots$ converges in probability. Let $F\colon \mathbf{R} \to \mathbf{R}$ be continuous. Show that $F(X_1), F(X_2), \ldots$ converges in probability to $F(X)$. More generally, suppose $\forall$ $1 \leq j \leq k$, $X_1^{(j)}, X_2^{(j)}, \ldots \colon \Omega \to \mathbf{R}$ is a sequence of random variables that converge in probability to $X^{(j)}$. Let $F\colon \mathbf{R}^k \to \mathbf{R}$ be continuous. Show that $F(X_i^{(1)}, \ldots, X_i^{(k)})$ converges in probability to $F(X^{(1)}, \ldots, X^{(k)})$. For example, if $k = 2$, then $X_1^{(1)} + X_1^{(2)}, X_1^{(2)} + X_2^{(2)}, \ldots$ converges in probability to $X^{(1)} + X^{(2)}$, and $X_1^{(1)} \cdot X_1^{(2)}, X_1^{(2)} \cdot X_2^{(2)}, \ldots$ converges in probability to $X^{(1)} \cdot X^{(2)}$.

(v) (Fatou's lemma for convergence in probability) If $X_1, X_2, \ldots \colon \Omega \to [0, \infty)$ converges in probability to X, show that $\mathbf{E}X \leq \liminf_{n\to\infty} \mathbf{E}X_n$.

(vi) (Dominated convergence in probability) If $X_1, X_2, \ldots$ converge in probability to X, and there exists a random variable $Y\colon \Omega \to [0, \infty)$ such that, for any $n \geq 1$, $|X_n| \leq Y$ and $\mathbf{E}Y < \infty$, then $\lim_{n\to\infty} \mathbf{E}X_n = \mathbf{E}X$.

Exercise 6. Let $X_1, \ldots, X_n\colon \Omega \to \mathbf{R}$ be uncorrelated random variables with $\mathbf{E}X_i^2 < \infty$ for any $1 \leq i \leq n$. Show that

$$\mathrm{var}(\sum_{i=1}^n X_i) = \sum_{i=1}^n \mathrm{var}(X_i)$$

Exercise 7 (L_2 Weak Law)**.** Let $\mu, c \in \mathbf{R}$. Let $X_1, X_2, \ldots \colon \Omega \to \mathbf{R}$ be uncorrelated random variables with $\mathbf{E}X_i = \mu$ and $\mathrm{var}(X_i) \leq c$ for all $i \geq 1$. Then $\frac{X_1 + \cdots + X_n}{n}$ converges to μ in L_2 as $n \to \infty$. So, $\frac{X_1 + \cdots + X_n}{n}$ converges to μ in probability as $n \to \infty$.

Exercise 8. A random variable $X\colon \Omega \to \mathbf{R}$ is said to be in weak L_1 if

$$\sup_{t>0} t\,\mathbf{P}(|X| > t) < \infty.$$

For example, a Cauchy distributed random variable X has density $f(x) = \frac{1}{\pi(1+x^2)}$ for any $x \in \mathbf{R}$, and X is in weak L_1 while $\mathbf{E}\,|X| = \infty$.

Show that, if $X_1, X_2, \ldots \colon \Omega \to (0, \infty)$ are i.i.d. such that X_1 is in weak L_1, then there exist real numbers $a_1, a_2, \ldots$ such that $\lim_{n\to\infty} a_n = \infty$ such that $\frac{1}{a_n}(X_1 + \cdots + X_n)$ converges in probability to 1.

(Hint: If you want to build up your intuition, assume $\mathbf{P}(X_1 > t) = 1/t$ for all $t > 2$, and use $b_n := n \log n$ in the Weak Law for Triangular Arrays.)

(Hint: Let $f(s) := \mathbf{E}X_1 1_{X_1 \leq s}$ for any $s > 0$. Note that $f(s)/s = \int_0^s (1/s)P(X > t)dt = \int_0^1 P(X > sx)dx \to 0$ as $s \to \infty$ by the Bounded Convergence Theorem. Choose $b_1 > b_2 > \cdots$ going to infinity such that $nf(b_n) = b_n$ for all large $n \geq 1$ as follows. When n is fixed and large, $nf(s)/s$ is larger than 1, and it converges to 0 as $s \to \infty$. Also, $nf(s)/s$ is right continuous in s, so let $b_n := \sup\{s > 0\colon nf(s)/s \leq 1\}$. Assume $\mathbf{E}X_1 = \infty$. Note that $\lim_{s\to\infty} \frac{f(s)}{s\mathbf{P}(X_1 > s)} = \infty$, so $\infty = \lim_{n\to\infty} \frac{f(b_n)}{b_n \mathbf{P}(X_1 > b_n)} = \lim_{n\to\infty} \frac{1}{n\mathbf{P}(X_1 > b_n)}$, i.e. $\lim_{n\to\infty} n\mathbf{P}(X_1 > b_n) = 0$. Now, use the Weak Law for Triangular arrays. Note that $\lim_{n\to\infty} \frac{b_n}{n} = \lim_{n\to\infty} f(b_n) = \lim_{s\to\infty} f(s) = \infty$, using $\mathbf{E}X_1 = \infty$.)