Please provide complete and well-written solutions to the following exercises.

Due February 1, at the beginning of class.

Homework 2

Exercise 1. Let $(\Omega,\mathcal{F},\mathbf{P})$ be a finite or countable probability space. If $X\colon\Omega\to[0,\infty]$ is a random variable with $\mathbf{E}\left|X\right|<\infty$, show that

$$\mathbf{E}X=\sum_{\omega\in\Omega}X(\omega)\mathbf{P}(\omega).$$

Exercise 2. Let X,Y be random variables such that $X,Y\geq 0$ or $\mathbf{E}\left|X\right|,\mathbf{E}\left|Y\right|<\infty$. Show:

- $\mathbf{E}(X+Y)=\mathbf{E}X+\mathbf{E}Y$ and if $c\in\mathbf{R}$, then $\mathbf{E}(cX)=c\mathbf{E}X$.
- If $\mathbf{P}(X=Y)=1$, then $\mathbf{E}X=\mathbf{E}Y$.
- $\mathbf{E}\left|X\right|\geq 0$ with equality only when $X=0$ almost surely.
- If $X\leq Y$ almost surely, then $\mathbf{E}X\leq\mathbf{E}Y$.

Exercise 3 (Inclusion-Exclusion Formula)**.** Let $A_1,\ldots,A_n\subseteq\Omega$ be events. Then:

$$\mathbf{P}(\cup_{i=1}^n A_i)=\sum_{i=1}^n\mathbf{P}(A_i)-\sum_{1\leq i<j\leq n}\mathbf{P}(A_i\cap A_j)+\sum_{1\leq i<j<k\leq n}\mathbf{P}(A_i\cap A_j\cap A_k)$$
$$\cdots+(-1)^{n+1}\mathbf{P}(A_1\cap\cdots\cap A_n).$$

To prove this formula, show that $1_{\cup_{i=1}^n A_i}=1-\prod_{i=1}^n(1-1_{A_i})$ and then take expected values of both sides.

Exercise 4. Let $\phi\colon\mathbf{R}\to\mathbf{R}$. We say that ϕ is **convex** if, for any $x,y\in\mathbf{R}$ and for any $t\in[0,1]$, we have

$$\phi(tx+(1-t)y)\leq t\phi(x)+(1-t)\phi(y).$$

Let $\phi\colon\mathbf{R}\to\mathbf{R}$. Show that ϕ is convex if and only if: for any $y\in\mathbf{R}$, there exists a constant a and there exists a function $L\colon\mathbf{R}\to\mathbf{R}$ defined by $L(x)=a(x-y)+\phi(y)$, $x\in\mathbf{R}$, such that $L(y)=\phi(y)$ and such that $L(x)\leq\phi(x)$ for all $x\in\mathbf{R}$. (In the case that ϕ is differentiable, the latter condition says that ϕ lies above all of its tangent lines.)

(Hint: Suppose ϕ is convex. If x is fixed and y varies, show that $\frac{\phi(y)-\phi(x)}{y-x}$ increases as y increases. Draw a picture. What slope a should L have at x?)

Exercise 5 (Jensen's Inequality)**.** Let $X\colon\Omega\to[-\infty,\infty]$ be a random variable. Let $\phi\colon\mathbf{R}\to\mathbf{R}$ be convex. Assume that $\mathbf{E}\left|X\right|<\infty$ and $\mathbf{E}\left|\phi(X)\right|<\infty$. Then

$$\phi(\mathbf{E}X)\leq\mathbf{E}\phi(X).$$

(Hint: use Exercise 4 with $y:=\mathbf{E}X$.) Deduce the **triangle inequality**:

$$\left|\mathbf{E}X\right|\leq\mathbf{E}\left|X\right|.$$

Exercise 6 (Markov's Inequality)**.** Let $X\colon \Omega \to [-\infty, \infty]$ be a random variable. Then

$$\mathbf{P}(|X| \geq t) \leq \frac{\mathbf{E}\,|X|}{t}, \qquad \forall\, t > 0.$$

(Hint: multiply both sides by t and use monotonicity of $\mathbf{E}$.)

Exercise 7. Combining Jensen's Inequality with the Monotone Convergence Theorem, show that if $\mathbf{E}X^2 < \infty$, then $\mathbf{E}\,|X| < \infty$, so $\mathbf{E}X \in \mathbf{R}$.

Exercise 8. Let $a, b \in \mathbf{R}$ and let $X\colon \Omega \to [-\infty, \infty]$ be a random variable with $\mathbf{E}X^2 < \infty$. Show that

$$\mathrm{var}(aX + b) = a^2 \mathrm{var}(X).$$

Then, let X be a standard Gaussian. Show that $\mathbf{E}X = 0$ and $\mathrm{var}(X) = 1$.

Finally, show that the quantity $\mathbf{E}(X - t)^2$ is minimized for $t \in \mathbf{R}$ uniquely when $t = \mathbf{E}X$.

Exercise 9 (The Chernoff Bound)**.** Let $X\colon \Omega \to [-\infty, \infty]$ be a random variable. Show that, for any $r, t > 0$,

$$\mathbf{P}(X > t) \leq e^{-rt}\mathbf{E}e^{rX}.$$

Exercise 10. Let $X, Y\colon \Omega \to \mathbf{R}$ be random variables. Let $0 < p < 1$ and let $||X||_p := (\mathbf{E}\,|X|^p)^{1/p}$. Show that there exists $c(p) > 0$ such that $||X + Y||_p \leq c(p)(||X||_p + ||Y||_p)$. In particular, it suffices to choose $c(p) = 2^{1/p}$. (Hint: a pointwise inequality should imply that $||X + Y||_p^p \leq ||X||_p^p + ||Y||_p^p$.)

Exercise 11. Let $X\colon \Omega \to [-\infty, \infty]$ be a random variable. Show that the function $p \mapsto ||X||_p$ is nondecreasing on the domain $p \in (0, \infty]$. So, if $||X||_p$ is finite for some value of p, then it is finite for all smaller values of p. (Hint: approximate X by bounded random variables, and then by apply the Monotone Convergence Theorem.)

Exercise 12 (Paley-Zygmund Inequality)**.** Let X be a nonnegative random variable with $\mathbf{E}X^2 < \infty$. Let $0 \leq t \leq 1$. Then

$$\mathbf{P}(X > t\,\mathbf{E}X) \geq (1 - t)^2 \frac{(\mathbf{E}X)^2}{\mathbf{E}X^2}.$$

(Hint: Apply the Cauchy-Schwarz inequality to $X1_{\{X > t\mathbf{E}X\}}$.)

Exercise 13 (Logarithmic Convexity of L_p-Norms)**.** Let X be a real-valued random variable. Let $0 < p_1 < p < p_2 \leq \infty$, and define $0 \leq t \leq 1$ by $\frac{1}{p} = \frac{1-t}{p_1} + \frac{t}{p_2}$. Then

$$||X||_p \leq ||X||_{p_1}^{(1-t)} ||X||_{p_2}^{t}$$