Please provide complete and well-written solutions to the following exercises.

Due April 26, at the beginning of class.

# Homework 12

**Exercise 1.** Let $x_1, x_2, \ldots$ be a sequence of real numbers. Show that total number of up-crossings of the sequence across the interval $[a,b]$ is finite for any $a, b \in \mathbf{R}$ with $a < b$ if and only if the sequence $x_1, x_2, \ldots$ converges to some $x \in [-\infty, \infty]$. (Here the random variables in the definition of up-crossing are chosen to be constant $X_m := x_m$ for all $m \geq 0$.)

**Exercise 2.** Below, we will use Doob's Up-crossing inequality to show that the $U[a,b]$ is finite almost surely for a nonnegative supermartingale. In this exercise, we derive Dubins' up-crossing inequality, an improvement to Doob's result that gives exponential decay of $\mathbf{P}(U[a,b] > t)$.

Let $((X_n^1)_{n\geq 0}, (\mathcal{F}_n)_{n\geq 0})$ and $((X_n^2)_{n\geq 0}, (\mathcal{F}_n)_{n\geq 0})$ be supermartingales and let $N$ be a stopping time such that $X_N^1 \geq X_N^2$.

- (i) (Switching Principle) For any $n \geq 0$, define $Y_n := X_n^1 1_{N>n} + X_n^2 1_{N\leq n}$. Show that $((Y_n)_{n\geq 0}, (\mathcal{F}_n)_{n\geq 0})$ is a supermartingale. Show the same conclusion for $Z_n := X_n^1 1_{N\geq n} + X_n^2 1_{N<n}$.

  That is, if we use $N$ to "switch" from one supermartingale to another, the random variables do not increase at the time of "switching", then we still have a supermartingale.
- (ii) Let $X_0, X_1, \ldots$ be a supermartingale with $X_n \geq 0$ for all $n \geq 0$. Let $a, b \in \mathbf{R}$ with $b > a > 0$. Let $N_0 := -1$ and for any integer $k \geq 1$, define

  $$N_{2k-1} := \min\{m > N_{2k-2} \colon X_m \leq a\}, \qquad N_{2k} := \min\{m > N_{2k-1} \colon X_m \geq b\}.$$

  Define $V_0, V_1, \ldots$ such that $V_n := 1$ for all $0 \leq n < N_1$, and for any $k \geq 1$,

  $$V_n := \begin{cases} (b/a)^{k-1}(X_n/a) & \text{, if } N_{2k-1} \leq n < N_{2k} \\ (b/a)^k & \text{, if } N_{2k} \leq n < N_{2k+1}. \end{cases}$$

  Using the switching principle, show by induction on $k$ that for any integer $k \geq 1$, $V_{0\wedge N_k}, V_{1\wedge N_k}, \ldots$ is a supermartingale.
- (iii) (Dubins' Inequality) Show that for any $b > a > 0$ and for any integer $t \geq 1$,

  $$\mathbf{P}(U[a,b] \geq t) \leq (a/b)^t \mathbf{E}\min(X_0/a, 1).$$

**Exercise 3.** Let $c_1, c_2, \ldots$ be positive constants such that $\sum_{n=1}^{\infty} c_n < \infty$. Let $Y_1, Y_2, \ldots$ be independent random variables such that $||Y_n||_\infty \leq c_n$ and $\mathbf{E}Y_n = 0$ for all $n \geq 1$, and let $X_n := Y_1 + \cdots + Y_n$ for all $n \geq 1$ with $X_0 := 0$. Show that $X_0, X_1, \ldots$ is a martingale that converges in every $L_p$ space with $1 \leq p < \infty$. That is, show there exists a real-valued random variable $X$ such that $\lim_{n\to\infty} ||X_n - X||_p = 0$.

**Exercise 4.** Let $H$ be a collection of random variables on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Let $Y\colon \Omega \to [0, \infty)$ with $\mathbf{E}Y < \infty$. Assume that, for all $X \in H$, $|X| \leq Y$. Show that $H$ is uniformly integrable. In particular, if $H$ is any finite set of random variables in $L_1$, then $H$ is uniformly integrable.

**Exercise 5.** Let $H$ be a collection of random variables on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Show that $H$ is uniformly integrable if and only if the following two conditions hold.

(a) $\sup_{X \in H} \mathbf{E}\,|X| < \infty$.
(b) For any $\varepsilon > 0$, there exists $\delta > 0$ such that
$$\sup\{\mathbf{E}\,|X|\,1_A\colon A \in \mathcal{F},\ \mathbf{P}(A) < \delta,\ X \in H\} < \varepsilon.$$

(Hint: when $X \in H$ is fixed, which $A$ with $\mathbf{P}(A) < \delta$ maximizes $\mathbf{E}\,|X|\,1_A$? Also, to show the first item, let $\varepsilon = 1/2$ in the definition of uniform integrability.)

**Exercise 6.** Let $H$ be a collection of random variables on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$. The analytic definition of uniform integrability is just the second item of the above exercise. That is, $H$ is uniformly integrable in the analytic sense if and only if condition (b) holds in Exercise 5. In the case that $\mathbf{P}$ is non-atomic, show that if condition (b) holds in Exercise 5, then condition (a) holds. In summary, if $\mathbf{P}$ is non-atomic, then the probabilistic and analytic definitions of uniform integrability coincide. (We say that $\mathbf{P}$ is non-atomic if for any $A \in \mathcal{F}$ with $\mathbf{P}(A) > 0$ there exists $B \in \mathcal{F}$ with $B \subseteq A$ such that $\mathbf{P}(A) > \mathbf{P}(B) > 0$. An atom for $\mathbf{P}$ is a set $A \in \mathcal{F}$ such that, for any $B \in \mathcal{F}$ with $B \subseteq A$, if $\mathbf{P}(B) < \mathbf{P}(A)$, then $\mathbf{P}(B) = 0$. In a non-atomic probability space, the following holds and you do not have to prove it: for any $A \in \mathcal{F}$ with $\mathbf{P}(A) > 0$, and for any $t \in \mathbf{R}$ such that $0 < t < \mathbf{P}(A)$, there exists $B \in \mathcal{F}$ with $B \subseteq A$ and $\mathbf{P}(B) = t$.)

**Exercise 7.** Show that condition (a) of Exercise 5 is not sufficient to prove uniform integrability. In fact, the unit ball $\{X \in L_1\colon \|X\|_1 \leq 1\}$ of $L_1$ is not uniformly integrable, in general. More specifically, find a set of random variables $X_1, X_2, \ldots$ on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with $\|X_n\|_1 \leq 1$ for all $n \geq 1$, but such that the collection $H := \{X_1, X_2, \ldots\}$ is not uniformly integrable. (Hint: let $\Omega := \{1, 2, 3, \ldots\}$ with the probability measure $\mathbf{P}$ defined by $\mathbf{P}(\{n\}) := 2^{-n}$ for all $n \geq 1$, and choose the $X_n$ to have disjoint supports.)

Now, let $p > 1$. Show that the unit ball $\{X \in L_p\colon \|X\|_p \leq 1\}$ of $L_p$ is uniformly integrable.

**Exercise 8.** Let $X_1, X_2, \ldots\colon \Omega \to \mathbf{R}$ be i.i.d. random variables with $\mathbf{E}\,|X_1| < \infty$. Then $\frac{X_1 + \cdots + X_n}{n}$ converges in $L_1$ to $\mathbf{E}X_1$.

(Hint: From the Strong Law, we already know that $\frac{X_1 + \cdots + X_n}{n}$ converges almost surely to $\mathbf{E}X_1$. So, conclude using the Vitali Convergence Theorem.)

**Exercise 9** (Galton-Watson Process)**.** Let $(\xi_{i,n})_{i,n \geq 1}$ be i.i.d. nonnegative integer-valued random variables. Let $Z_0 := 1$ and for any $n \geq 0$ define
$$Z_{n+1} := \begin{cases} \xi_{1,n+1} + \cdots + \xi_{Z_n,n+1} & \text{, if } Z_n > 0 \\ 0 & \text{, if } Z_n = 0. \end{cases}$$
Then $Z_0, Z_1, \ldots$ is an example of a branching process, known as the Galton-Watson process. The intuition is that $Z_n$ is the number of individuals in the $n^{th}$ generation of a family tree,

and at each time step, each person has a certain number of offspring. Galton and Watson originally used this process to model the occurrence of last names in human family trees, to see why some names become common while others become extinct.

$\forall\, n \geq 0$, let $\mathcal{F}_n := \sigma(\xi_{i,m} \colon i \geq 1, 1 \leq m \leq n)$, and let $\mu := \mathbf{E}\xi_{1,1}$. Assume $\mu \in (0, \infty)$.

- Show that $Z_0, Z_1/\mu, Z_2/\mu^2, \ldots$ is a martingale with respect to $\mathcal{F}_0, \mathcal{F}_1, \ldots$. (Hint: write $\mathbf{E}(Z_{n+1}|\mathcal{F}_n) = \sum_{k=1}^{\infty} \mathbf{E}(Z_{n+1} 1_{Z_n=k}|\mathcal{F}_n)$.)
- If $\mu < 1$, show that $\mathbf{P}(Z_n > 0$ for infinitely many $n \geq 0) = 0$. Therefore, $Z_n/\mu^n$ converges to 0 almost surely as $n \to \infty$. Also, show that the expected total population $\mathbf{E}\sum_{n=0}^{\infty} Z_n$ is finite. That is, extinction occurs as $n \to \infty$ if the average number of offspring is less than 1 for each individual.
- If $\mu = 1$, and $\mathbf{P}(\xi_{1,1} = 1) < 1$, show that $Z_0, Z_1, \ldots$ converges almost surely to 0.

**Exercise 10.** Explain why the following example does not contradict Doob's Optional Stopping Theorem.

Let $X_0, X_1, \ldots$ be the simple random walk on $\mathbf{Z}$. Note that $\mathbf{E}X_0 = 0$ and $X_0, X_1, \ldots$ is a martingale. Let $N := \min\{n \geq 1 \colon X_n = 1\}$ be the return time to 1. Then $N$ is a stopping time and $X_N = 1$, so $\mathbf{E}X_N = 1 \neq 0 = \mathbf{E}X_0$.

**Exercise 11** (Gambler's Ruin)**.** We can now finally answer the question posed in our introduction to the Gambler's Ruin Problem. Let $0 < p < 1$. Let $0 \leq a < y_0 < b$ with $a, y_0, b \in \mathbf{Z}$. Let $Y_1, Y_2, \ldots$ be independent random variables such that $\mathbf{P}(Y_n = 1) =: p$ and $\mathbf{P}(Y_n = -1) = 1-p =: q\ \forall\, n \geq 1$. Let $Y_0 := y_0$. Let $Z_n = Y_0 + \cdots + Y_n$, and let $X_n := (q/p)^{Z_n}$ $\forall\, n \geq 0$. For any $n \geq 0$, let $\mathcal{F}_n := \sigma(Y_0, \ldots, Y_n)$. We showed in class that $X_0, X_1, \ldots$ is a martingale with respect to $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots$. Let $T = \min\{n \geq 1 \colon Z_n \in \{a, b\}\}$. That is, $T$ is the first time the random walk $Z_0, Z_1, \ldots$ hits either $a$ or $b$.

- Compute $c := \mathbf{P}(Y_T = a)$, using Doob's Optional Stopping Theorem, when $p \neq 1/2$.
- Compute $\mathbf{E}T$ using Doob's Optional Stopping Theorem, when $p \neq 1/2$. (Hint: $Z_0 - 0(2p-1), Z_1 - (2p-1), \ldots$ is a martingale.)
- Compute $\mathbf{E}\min\{n \geq 1 \colon Z_n = a\}$ when $p < 1/2$. (Hint: let $b \to \infty$.)
- Compute $c$ when $p = 1/2$ using the martingale $Z_0, Z_1, \ldots$.
- Compute $\mathbf{E}T$ when $p = 1/2$. (Hint: if $y_0 = 0$, then $Z_0^2 - 0, Z_1^2 - 1, \ldots$ is a martingale.)