Please provide complete and well-written solutions to the following exercises.

Due April 19, at the beginning of class.

Homework 11

Exercise 1 (Binomial Option Pricing Model)**.** Let $u,d>0$. Let $0<p<1$. Let $Y_1,Y_2,\ldots$ be independent random variables such that $\mathbf{P}(Y_n=\log u)=:p$ and $\mathbf{P}(Y_n=\log d)=1-p$ $\forall\, n\geq 1$. Let Z_0 be a fixed constant. Let $Z_n:=Y_0+\cdots+Y_n$, and let $V_n:=e^{Z_n}$ $\forall\, n\geq 1$. In general, $V_0,V_1,\ldots$ will not be a martingale, but we can e.g. compute $\mathbf{E}V_n$, by modifying $V_0,V_1,\ldots$ to be a martingale.

First, note that if $n\geq 1$, then Z_n has a binomial distribution, in the sense that

$$\mathbf{P}(Z_n=X_0+i\log u+(n-i)\log d)=\binom{n}{i}p^i(1-p)^{n-i},\qquad \forall\, 0\leq i\leq n.$$

For any $n\geq 1$, let $\mathcal{F}_n:=\sigma(Y_0,\ldots,Y_n)$. Define

$$r:=p(u-d)-1+d.$$

Here we chose r so that $p=\frac{1+r-d}{u-d}$. For any $n\geq 0$, define

$$X_n:=(1+r)^{-n}V_n.$$

Show that $X_0,X_1,\ldots$ is a martingale with respect to $\mathcal{F}_0\subseteq\mathcal{F}_1\subseteq\cdots$. Consequently,

$$(1+r)^{-n}\mathbf{E}S_n=\mathbf{E}S_0,\qquad \forall\, n\geq 0.$$

Exercise 2. Let $M_0,M_1,\ldots$ be a martingale with $\mathbf{E}M_n^2<\infty$ for all $n\geq 0$. Show that the increments $M_2-M_1,M_3-M_2,\ldots$ are orthogonal in the following sense. For any $i,j\geq 1$ with $i\neq j$,

$$\mathbf{E}(M_{i+1}-M_i)(M_{j+1}-M_j)=0.$$

This property is sometimes called **orthogonality of martingale increments**.

Exercise 3. Let X be a real-valued random variable on a probability space $(\Omega,\mathcal{F},\mathbf{P})$. Assume $\mathbf{E}\,|X|<\infty$. Let $\mathcal{F}_0\subseteq\mathcal{F}_1\subseteq\cdots\mathcal{F}$ be σ-algebras. For any $n\geq 0$, define $X_n:=\mathbf{E}(X|\mathcal{F}_n)$. Show that $X_0,X_1,\ldots$ is a martingale. (Optional challenge question: For any martingale $((X_n)_{n\geq 0},(\mathcal{F}_n)_{n\geq 0})$, is there a random variable X with $\mathbf{E}\,|X|<\infty$ such that $X_n=\mathbf{E}(X|\mathcal{F}_n)$ for all $n\geq 0$?)

Exercise 4. Let M,N be stopping times for a martingale $((X_n)_{n\geq 0},(\mathcal{F}_n)_{n\geq 0})$. Show that $\max(M,N)$ and $\min(M,N)$ are stopping times. In particular, if $n\geq 0$ is fixed, then $\max(M,n)$ and $\min(M,n)$ are stopping times

Exercise 5. Let $X_0,X_1,\ldots$ and let $Y_0,Y_1,\ldots$ be submartingales adapted to the same filtration $\mathcal{F}_0\subseteq\mathcal{F}_1\subseteq\cdots$. Show that $X_0+Y_0,X_1+Y_1,\ldots$ is a submartingale adapted to the filtration $\mathcal{F}_0\subseteq\mathcal{F}_1\subseteq\cdots$. Consequently, a sum of supermartingales is a supermartingale, and a sum of martingales is a martingale (when they are adapted to the same filtration).

Exercise 6.

(i) Let $((X_n)_{n\geq 0}, (\mathcal{F}_n)_{n\geq 0})$ be a submartingale. Show that, almost surely, $\mathbf{E}(X_n|\mathcal{F}_m) \geq X_m$ for any $n > m$. Consequently, $n \mapsto \mathbf{E}X_n$ is nondecreasing.
(ii) Let $((X_n)_{n\geq 0}, (\mathcal{F}_n)_{n\geq 0})$ be a supermartingale. Show that, almost surely, $\mathbf{E}(X_n|\mathcal{F}_m) \leq X_m$ for any $n > m$. Consequently, $n \mapsto \mathbf{E}X_n$ is nonincreasing.
(iii) Let $((X_n)_{n\geq 0}, (\mathcal{F}_n)_{n\geq 0})$ be a martingale. Let $\phi\colon \mathbf{R} \to \mathbf{R}$ be convex. Assume $\mathbf{E}\,|\phi(X_n)| < \infty$ for all $n \geq 1$. Show that $((\phi(X_n))_{n\geq 0}, (\mathcal{F}_n)_{n\geq 0})$ is a submartingale.
(iv) Let $((X_n)_{n\geq 0}, (\mathcal{F}_n)_{n\geq 0})$ be a submartingale. Let $\phi\colon \mathbf{R} \to \mathbf{R}$ be convex and nondecreasing. Assume $\mathbf{E}\,|\phi(X_n)| < \infty$ for all $n \geq 1$. Show that $((\phi(X_n))_{n\geq 0}, (\mathcal{F}_n)_{n\geq 0})$ is a submartingale.

Exercise 7 (Azuma's Inequality)**.** In this exercise, we prove a generalization of the Hoeffding inequality to martingales. Let $c_1, c_2, \ldots > 0$. Let $((X_n)_{n\geq 0}, (\mathcal{F}_n)_{n\geq 0})$ be a martingale. Assume that $|X_n - X_{n-1}| \leq c_n$ for all $n \geq 1$. Then for any $t > 0$,

$$\mathbf{P}(|X_n - X_0| > t) \leq 2e^{-\frac{t^2}{2\sum_{i=1}^n c_i^2}}.$$

Prove this inequality using the following steps.

- Let $\alpha > 0$. Show that $\mathbf{E}e^{\alpha(X_n - X_0)} = \mathbf{E}[e^{\alpha(X_{n-1}-X_0)}\mathbf{E}(e^{\alpha(X_n - X_{n-1})}|\mathcal{F}_{n-1})]$.
- For any $y \in [-1, 1]$, show that $e^{\alpha c_n y} \leq \frac{1+y}{2}e^{\alpha c_n} + \frac{1-y}{2}e^{-\alpha c_n}$.
- Take the conditional expectation of this inequality when $y = (X_n - X_{n-1})/c_n$.
- Now argue as in Hoeffding's inequality.

Using Azuma's inequality, deduce **McDiarmid's Inequality**. Let $X_1, \ldots, X_n$ be independent real-valued random variables. Let $c_1, c_2, \ldots > 0$. Let $f\colon \mathbf{R}^n \to \mathbf{R}$ be a measurable function such that, for any $1 \leq m \leq n$,

$$\sup_{x_1,\ldots,x_{m-1},x_m,x_m',x_{m+1},\ldots,x_n \in \mathbf{R}} |f(x_1, \ldots, x_n) - f(x_1, \ldots, x_{m-1}, x_m', x_{m+1}, \ldots, x_n)| \leq c_m.$$

Then, for any $t > 0$,

$$\mathbf{P}(|f(X_1, \ldots, X_n) - \mathbf{E}f(X_1, \ldots, X_n)| > t) \leq 2e^{-\frac{t^2}{2\sum_{i=1}^n c_i^2}}.$$

(Note that a linear function f recovers Hoeffding's inequality.)

Exercise 8. In the L_p maximal inequality, the constant $\frac{p}{p-1}$ goes to infinity as $p \to 1^+$. So, one might guess that the L_p maximal inequality does not hold when $p = 1$. (If so, this justifies the need to prove a weaker statement when $p = 1$, i.e. Doob's inequality.) Using the simple random walk on $\mathbf{Z}$, show that the L_p maximal inequality does not hold when $p = 1$. (Hint: use the probabilities for a simple random walk taking one value before another, as we did in class after Wald's equation was introduced.)

Exercise 9. Show that the second part of the L_p maximal inequality cannot hold when $X_0, X_1, \ldots$ is a submartingale. That is, for any $n \geq 1$, find a submartingale $X_0, X_1, \ldots$ such that, for any $p > 1$, $||\max_{0\leq m\leq n} |X_m|||_p > 0$ but such that $||X_n||_p = 0$. (Hint: just consider a non-random sequence of numbers.)