Please provide complete and well-written solutions to the following exercises.

Due April 12, at the beginning of class.

Homework 10

Exercise 1. Prove Wald's first equation. Let $X_1, X_2, \ldots : \Omega \to \mathbf{R}$ be i.i.d. with $\mathbf{E}\,|X_1| < \infty$. Let N be a stopping time with $\mathbf{E}N < \infty$. Let $S_0 := 0$ and for any $n \geq 1$, let $S_n := X_1 + \cdots + X_n$. Then $\mathbf{E}S_N = \mathbf{E}X_1\mathbf{E}N$. (Hint: condition on N taking fixed values.)

Exercise 2. Let $\Omega = [0,1]$. Let $\mathbf{P}$ be the uniform probability law on Ω. Let $X\colon [0,1] \to \mathbf{R}$ be a random variable such that $X(t) = t^2$ for all $t \in [0,1]$. Let

$$\mathcal{G} = \sigma\{[0,1/4),[1/4,1/2),[1/2,3/4),[3/4,1]\}.$$

Compute explicitly the function $\mathbf{E}(X|\mathcal{G})$. (It should be constant on each of the partition elements.) Draw the function $\mathbf{E}(X|\mathcal{G})$ and compare it to a drawing of X itself.

Now, for every integer $k > 1$, let $s = 2^{-k}$, and let $\mathcal{G}_k := \sigma\{[0,s),[s,2s),[2s,3s),\ldots,[1-2s,1-s),[1-s,1)\}$. Try to draw $\mathbf{E}(X|\mathcal{G}_k)$. Prove that, for every $t \in [0,1]$,

$$\lim_{k\to\infty} \mathbf{E}(X|\mathcal{G}_k)(t) = X(t).$$

Exercise 3. Let $X\colon \Omega \to \mathbf{R}$ be a random variable with finite variance, and let $t \in \mathbf{R}$. Consider the function $f\colon \mathbf{R} \to \mathbf{R}$ defined by $f(t) = \mathbf{E}(X-t)^2$. Show that the function f is uniquely minimized when $t = \mathbf{E}X$. That is, $f(\mathbf{E}X) < f(t)$ for all $t \in \mathbf{R}$ such that $t \neq \mathbf{E}X$. Put another way, setting t to be the mean of X minimizes the quantity $\mathbf{E}(X-t)^2$ uniquely.

The conditional expectation, being a piecewise version of taking an average, has a similar property. Let $B_1, \ldots, B_k \subseteq \Omega$ such that $B_i \cap B_j = \emptyset$ for all $i,j \in \{1,\ldots,k\}$ with $i \neq j$, and $\cup_{i=1}^k B_i = \Omega$. Write $\mathcal{G} = \sigma\{B_1,\ldots,B_k\}$. Let Y be any other random variable such that, for each $1 \leq i \leq k$, Y is constant on B_i. Show that the quantity $\mathbf{E}(X-Y)^2$ is uniquely minimized by such a Y only when $Y = \mathbf{E}(X|\mathcal{G})$.

Exercise 4. Let $\Omega = [0,1]$. Let $\mathbf{P}$ be the uniform probability law on Ω. Let $X\colon [0,1] \to \mathbf{R}$ be a random variable such that $X(t) = t^2$ for all $t \in [0,1]$. For every integer $k > 1$, let $s = 2^{-k}$, let $\mathcal{G}_k := \sigma\{[0,s),[s,2s),[2s,3s),\ldots,[1-2s,1-s),[1-s,1)\}$, and let $M_k := \mathbf{E}(X|\mathcal{G}_k)$. Show that the increments $M_2 - M_1, M_3 - M_2, \ldots$ are orthogonal in the following sense. For any $i,j \geq 1$ with $i \neq j$,

$$\mathbf{E}(M_{i+1} - M_i)(M_{j+1} - M_j) = 0.$$

This property is sometimes called **orthogonality of martingale increments**.

Exercise 5. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space, and let $X\colon \Omega \to \mathbf{R}$ be a random variable with $\mathbf{E}\,|X| < \infty$. Let $\mathcal{G}, \mathcal{H} \subseteq \mathcal{F}$ be σ-algebras. Let $\mathcal{H}$ be a σ-algebra that is independent of $\sigma(\sigma(X), \mathcal{G})$. Show that

$$\mathbf{E}(X|\sigma(\mathcal{G},\mathcal{H})) = \mathbf{E}(X|\mathcal{G}).$$

In particular, if we choose $\mathcal{G} = \{\emptyset, \Omega\}$, we get: if $\mathcal{H}$ is independent of $\sigma(X)$, then $\mathbf{E}(X|\mathcal{H}) = \mathbf{E}X$.

(Hint: Let $G \in \mathcal{G}, H \in \mathcal{H}$, let $Y := \mathbf{E}(X|\mathcal{G})$. Compare $\mathbf{E}(X1_{G\cap H})$ and $\mathbf{E}(Y1_{G\cap H})$. Is the set of $A \in \sigma(\mathcal{G}, \mathcal{H})$ such that $\mathbf{E}(X1_A) = \mathbf{E}(Y1_A)$ a monotone class?)

Exercise 6. Prove Jensen's inequality for the conditional expectation. Let $X\colon \Omega \to \mathbf{R}$ be a random variable and let $\phi\colon \mathbf{R} \to \mathbf{R}$ be convex. Assume $\mathbf{E}\,|X|\,, \mathbf{E}\,|\phi(X)| < \infty$. Then

$$\phi(\mathbf{E}(X|\mathcal{G})) \leq \mathbf{E}(\phi(X)|\mathcal{G})$$

Conclude that for any $1 \leq p \leq \infty$ we have the following contractive inequality for conditional expectation

$$\|\mathbf{E}(X|\mathcal{G})\|_p \leq \|X\|_p\,.$$

Exercise 7 (Tower Property)**.** Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space, and let $X\colon \Omega \to \mathbf{R}$ be a random variable with $\mathbf{E}\,|X| < \infty$. Let $\mathcal{H} \subseteq \mathcal{G} \subseteq \mathcal{F}$ be σ-algebras. Then $\mathbf{E}(X|\mathcal{H}) = \mathbf{E}(\mathbf{E}(X|\mathcal{G})|\mathcal{H})$.

Exercise 8 (Conditional Markov Inequality)**.** Let $p > 0$. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space, and let $X\colon \Omega \to \mathbf{R}$ be a random variable with $\mathbf{E}\,|X|^p < \infty$. Let $\mathcal{G} \subseteq \mathcal{F}$ be a σ-algebra. For any $A \in \mathcal{F}$, we denote $\mathbf{P}(A|\mathcal{G}) := \mathbf{E}(1_A|\mathcal{G})$.

- Show that, almost surely,
$$\mathbf{E}(|X|^p\,|\mathcal{G}) = \int_0^\infty pt^{p-1}\mathbf{P}(|X| > t|\mathcal{G})dt.$$
- Deduce a conditional version of Markov's inequality: for any $t > 0$, almost surely,
$$\mathbf{P}(|X| > t|\mathcal{G}) \leq \frac{\mathbf{E}(|X|^p\,|\mathcal{G})}{t^p}.$$

Exercise 9 (Conditional Hölder Inequality)**.** Let $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space, and let $X, Y\colon \Omega \to \mathbf{R}$ be random variables with $\mathbf{E}\,|X|^p, \mathbf{E}\,|Y|^q < \infty$. Let $\mathcal{G} \subseteq \mathcal{F}$ be a σ-algebra. Show that, almost surely,

$$\mathbf{E}(|XY|\,|\mathcal{G}) \leq [\mathbf{E}(|X|^p\,|\mathcal{G})]^{1/p}[\mathbf{E}(|Y|^q\,|\mathcal{G})]^{1/q}.$$

Exercise 10. Let H be a Hilbert space. Let $g, h \in H$. Prove the Cauchy-Schwarz inequality

$$|\langle g, h\rangle| \leq \|g\|\,\|h\|\,.$$

Show also the triangle inequality $\|g + h\| \leq \|g\| + \|h\|$, and the parallelogram law $\|g + h\|^2 + \|g - h\|^2 = 2\,\|g\|^2 + 2\,\|h\|^2$.

Exercise 11. Let H be a Hilbert space, let $M \subseteq H$ a closed subspace, and for any $h \in H$, denote $f(h)$ as the linear projection of H onto M. Show that $h \mapsto f(h)$ is actually a linear projection. That is, verify that $f(\alpha g + h) = \alpha f(g) + h$ and $f(f(h)) = f(h)$ for any $\alpha \in \mathbf{R}$, $g, h \in H$.

Exercise 12. Let X be $\mathcal{F}$-measurable and let Y be $\mathcal{G}$-measurable, real-valued random variables, where $\mathcal{G} \subseteq \mathcal{F}$. Let $\mu_{X|\mathcal{G}}$ be a regular conditional probability on $\mathcal{F}$ given $\mathcal{G}$. Let

$h\colon \mathbf{R}^2 \to \mathbf{R}$ be a Borel measurable function with $\mathbf{E}\,|h(X,Y)| < \infty$. Then, almost surely with respect to $\omega \in \Omega$,

$$\mathbf{E}(h(X,Y)|\mathcal{G})(\omega) = \int_{\mathbf{R}} h(x, Y(\omega))\mu_{X|\mathcal{G}}(x,\omega)dx.$$

In particular, if Y is constant and if $\mathbf{E}\,|X| < \infty$,

$$\mathbf{E}(X|\mathcal{G})(\omega) = \int_{\mathbf{R}} x\mu_{X|\mathcal{G}}(x,\omega)dx.$$