60850 Final Solutions[1]

1. Question 1

This problem proves a monotone convergence theorem for conditional expectation.

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space. Let $0 \leq X_1 \leq X_2 \leq \cdots$ be $\mathcal{F}$-measurable random variables. Let $X := \lim_{n\to\infty} X_n$ be the pointwise limit of $X_1, X_2, \ldots$. Assume $\mathbf{E}X < \infty$. Let $\mathcal{G} \subseteq \mathcal{F}$ be a σ-algebra. Then

$$\lim_{n\to\infty} \mathbf{E}(X_n|\mathcal{G}) = \mathbf{E}(X|\mathcal{G}).$$

(Hint: first prove that the sequence $\mathbf{E}(X_1|\mathcal{G}), \mathbf{E}(X_2|\mathcal{G}), \ldots$ has a pointwise limit, almost surely.)

Solution. Since conditional expectation is monotone by Corollary 5.14, $0 \leq X_1 \leq X_2 \leq \cdots$ implies that $0 \leq \mathbf{E}(X_1|\mathcal{G}) \leq \mathbf{E}(X_2|\mathcal{G}) \leq \cdots$, so there exists some $\mathcal{G}$-measurable random variable Z such that $Z = \lim_{n\to\infty} \mathbf{E}(X_n|\mathcal{G})$ almost surely. Then for any $A \in \mathcal{G}$ we have

$$\begin{aligned}
\mathbf{E}Z1_A &= \mathbf{E}(\lim_{n\to\infty} \mathbf{E}(X_n|\mathcal{G}))1_A \qquad \text{, by definition of } Z\\
&= \lim_{n\to\infty} \mathbf{E}(\mathbf{E}(X_n|\mathcal{G}))1_A \qquad \text{, by the usual monotone convergence theorem}\\
&= \lim_{n\to\infty} \mathbf{E}X_n 1_A \qquad \text{, by the definition of conditional expectation}\\
&= \mathbf{E}X1_A \qquad \text{, by the usual monotone convergence theorem.}
\end{aligned}$$

Note that Z is $\mathcal{G}$ measurable, and by choosing $A := \Omega$, we see that $\mathbf{E}Z < \infty$. So, by the definition of conditional expectation, we have $Z = \mathbf{E}(X|\mathcal{G})$, as desired.

2. Question 2

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space. As usual, if $X \colon \Omega \to \mathbf{R}$ is measurable, define $\|X\|_1 := \mathbf{E}|X|$. Let H be an L_1 bounded set of random variables on $(\Omega, \mathcal{F}, \mathbf{P})$, i.e. assume that $\sup_{X\in H} \|X\|_1 < \infty$.

Suppose H is not uniformly integrable. Show that there exists $\varepsilon > 0$ and there exist disjoint sets $A_1, A_2, \ldots \subseteq \Omega$ such that, for all $n \geq 1$, we have

$$\sup_{X\in H} \mathbf{E}|X| 1_{A_n} \geq \varepsilon.$$

(You are not allowed to use without proof Theorem 6.60 cited in the Additional Comments section of the notes.)

(Hint: start by using Exercise 6.41 in the notes to get $\varepsilon > 0$. Let $\{\delta_{jk}\}_{1\leq j<k<\infty}$ be nonnegative real numbers such that $\sum_{1\leq j<k<\infty} \delta_{jk} < \varepsilon/2$. Next, find $X_1, X_2, \ldots \in H$ and sets $B_1, B_2, \ldots \in \mathcal{F}$ such that

$$\mathbf{E}|X_k| 1_{B_k} > \varepsilon \;\forall\; k \geq 1.$$

$$\mathbf{E}|X_j| 1_{B_k} < \delta_{jk} \;\forall\; 1 \leq j < k.)$$

Solution. Since H is not uniformly integrable and H is bounded, by Exercise 6.41 in the notes, there exists $\varepsilon > 0$ such that, for any $\delta > 0$, there exists $A_\delta \in \mathcal{F}$ and $X_\delta \in H$ with $\mathbf{P}(A_\delta) < \delta$ and $\mathbf{E}|X| 1_{A_\delta} > \varepsilon$. Let $\{\delta_{jk}\}_{1\leq j<k<\infty}$ be nonnegative real numbers such that

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$\sum_{1\leq j<k<\infty}\delta_{jk} < \varepsilon/2$. We claim we can inductively define a sequence $X_1, X_2, \ldots \in H$ and sets $B_1, B_2, \ldots \in \mathcal{F}$ such that

$$\mathbf{E}\,|X_k|\,1_{B_k} > \varepsilon \;\forall\; k \geq 1.$$

$$\mathbf{E}\,|X_j|\,1_{B_k} < \delta_{jk} \;\forall\; 1 \leq j < k.$$

Indeed, given $X_1, \ldots, X_k$ and $B_1, \ldots, B_k$, note that $X_1, \ldots, X_k$ is a finite set of random variables, hence it is uniformly integrable by Exercise 6.40 in the notes. So, by Exercise 6.41 in the notes, there exists $\delta > 0$ such that for any $B \in \mathcal{F}$ with $\mathbf{P}(B) < \delta$, we have $\mathbf{E}\,|X_j|\,1_B < \delta_{jk}$ for all $1 \leq j \leq n$. Then, since H is not uniformly integrable, there exists $B_{n+1} \in \mathcal{F}$ and a random variable $X_{n+1} \in H$ such that $\mathbf{P}(B_{n+1}) < \delta$ and $\mathbf{E}\,|X_{n+1}|\,1_{B_{n+1}} > \varepsilon$.

For any $j \geq 1$, let $A_j := B_j \smallsetminus \cup_{k>j} B_k$. Then $A_1, A_2, \ldots$ are disjoint and

$$\mathbf{E}\,|X_j|\,1_{A_j} \geq \mathbf{E}\,|X_j|\,1_{B_j} - \sum_{k>j}\mathbf{E}\,|X_j|\,1_{B_k} > \varepsilon - \sum_{k>j}\delta_{jk} > \varepsilon/2.$$

3. Question 3

Let $Y_1, Y_2, \ldots$ be independent random variables such that $\mathbf{P}(Y_n = 1) = \mathbf{P}(Y_n = -1) = 1/2$ $\forall\; n \geq 1$. Let $Y_0 := 0$. Let $Z_n := Y_0 + \cdots + Y_n$ for any $n \geq 0$. From the homework, one might wonder where the martingale $Z_0^2 - 0, Z_1^2 - 1, \ldots$ came from, and if more like it exist. In this exercise, we compute an infinite family of such martingales.

For any $\alpha \in \mathbf{R}$ and $n \geq 0$, let $X_n := e^{\alpha Z_n - n\log\cosh(\alpha)}$. Show that $X_0, X_1, \ldots$ is a martingale.

Then, using the power series expansion of the exponential function, we have $X_n = \sum_{m=0}^{\infty}\frac{\alpha^m}{m!}M_{m,n}$ for some random variables $M_{1,1}, \ldots$, for any $\alpha \in \mathbf{R}$ and for any $n \geq 0$. Show that it follows that, for any $m \geq 0$, $M_{m,0}, M_{m,1}, \ldots$ is a martingale. For example, using $m = 2$ we get $M_{2,n} = Z_n^2 - n$ for all $n \geq 0$. And using $m = 4$, $M_{4,n} = Z_n^4 - 6nZ_n^2 + 2n + 3n^2$ for all $n \geq 0$. (You can assume that this formula holds for $M_{4,n}$.) Using the martingale $(M_{4,n})_{n\geq 0}$, compute $\mathbf{E}T^2$ when $T := \min\{n \geq 1\colon Z_n \in \{-b, b\}\}$ and $b > 0$, $b \in \mathbf{Z}$.

Solution. Let $\mathcal{F}_n := \sigma(Y_0, \ldots, Y_n)$ for all $n \geq 0$. Then

$$\begin{aligned}
\mathbf{E}(X_{n+1} - X_n|\mathcal{F}_n) &= e^{-n\log\cosh(\alpha)}\mathbf{E}(e^{\alpha(Y_{n+1}+Z_n)-\log\cosh(\alpha)} - e^{\alpha Z_n}|\mathcal{F}_n)\\
&= e^{\alpha Z_n - n\log\cosh(\alpha)}\mathbf{E}(e^{\alpha Y_{n+1}-\log\cosh(\alpha)} - 1|\mathcal{F}_n) \quad \text{, since } Z_n \text{ is } \mathcal{F}_n\text{-measurable, and by Prop. 5.16}\\
&= X_n\mathbf{E}(e^{\alpha Y_{n+1}-\log\cosh(\alpha)} - 1) \qquad \text{, since } Y_{n+1} \text{ is independent of } \mathcal{F}_n\text{, and by Exercise 5.11}\\
&= X_n\Big(\frac{e^{\alpha}+e^{-\alpha}}{2\cosh(\alpha)} - 1\Big) = 0.
\end{aligned}$$

So, $X_1, X_2, \ldots$ is a martingale (each random variable is bounded, so $\mathbf{E}\,|X_n| < \infty$ for all $n \geq 1$, and X_n is $\mathcal{F}_n$-measurable by definition of X_n.)

By the power series expansion of the exponential (with respect to α) we then have $X_n = \sum_{m=0}^{\infty}\frac{\alpha^m}{m!}M_{m,n}$ for some random variables $M_{1,1}, \ldots$, for any $\alpha \in \mathbf{R}$. Since the martingale property holds for every $\alpha \in \mathbf{R}$, it follows that for any $m \geq 0$, $M_{m,0}, M_{m,1}, \ldots$ is a martingale. For example, to find the $m = 4$ term in the expansion, we take four derivatives in α and set $\alpha = 0$ to get $M_{4,n} = Z_n^4 - 6nZ_n^2 + 2n + 3n^2$ for all $n \geq 0$. Fix $n > 0$ and consider the stopping time $T \wedge n$. From the Optional Stopping Theorem, Version 1, $\mathbf{E}M_{4,T\wedge n} = \mathbf{E}M_{4,0} = 0$. That is,

$$0 = \mathbf{E}M_{4,0} = \mathbf{E}M_{4,T\wedge n} = \mathbf{E}Z_{T\wedge n}^4 - 6\mathbf{E}(T\wedge n)Z_{T\wedge n}^2 + 2\mathbf{E}T\wedge n + 3\mathbf{E}(T\wedge n)^2$$

That is, for any $n \geq 0$,

$$\mathbf{E}(T \wedge n)^2 = \frac{1}{3}(-\mathbf{E}Z_{T\wedge n}^4 + 6\mathbf{E}(T \wedge n)Z_{T\wedge n}^2 - 2\mathbf{E}T \wedge n)$$

We will treat each term on the right separately. As shown in the notes in Example 4.22, $\mathbf{E}T = b^2$ and $\mathbf{P}(T = \infty) = 0$, so by the Monotone Convergence Theorem,

$$\lim_{n\to\infty} \mathbf{E}T \wedge n = \mathbf{E}T = b^2.$$

Also

$$\left|\mathbf{E}(T \wedge n)Z_{T\wedge n}^2\right| \leq b^2\mathbf{E}(T \wedge n) \leq b^2\mathbf{E}T = b^4.$$

So, by the Dominated Convergence Theorem,

$$\lim_{n\to\infty} \mathbf{E}(T \wedge n)Z_{T\wedge n}^2 = \mathbf{E}TZ_T^2 = b^2\mathbf{E}T = b^4.$$

Finally, by the Bounded Convergence Theorem

$$\lim_{n\to\infty} \mathbf{E}Z_{T\wedge n}^2 = \mathbf{E}Z_T^2 = b^2.$$

Combining these facts and using the Monotone Convergence Theorem,

$$\mathbf{E}T^2 = \lim_{n\to\infty} \mathbf{E}(T \wedge n)^2 = \frac{1}{3}(-b^4 + 6b^4 - b^2) = \frac{b^2}{3}(5b^2 - 2).$$

4. Question 4

Let $0 < p < 1$. Let b be a positive integer. Let $Y_1, Y_2, \ldots$ be independent random variables such that $\mathbf{P}(Y_n = 1) =: p$ and $\mathbf{P}(Y_n = -1) = 1 - p =: q$ $\forall$ $n \geq 1$. Let $Y_0 := 0$. Let $Z_n = Y_0 + \cdots + Y_n$, $\forall$ $n \geq 0$. Let $T_b := \min\{n \geq 1 \colon Z_n = b\}$. For any $\alpha \in \mathbf{R}$ let $M(\alpha) := \mathbf{E}e^{\alpha Y_1}$. For any $n \geq 0$, let

$$X_n := e^{\alpha Z_n}(M(\alpha))^{-n}.$$

- If $1/2 \leq p < 1$, show that $e^{\alpha b}\mathbf{E}M(\alpha)^{-T_b} = 1$ for all $\alpha > 0$.
- If $p = 1/2$ and $0 < s < 1$, show that

$$\mathbf{E}s^{T_1} = \frac{s}{1 + \sqrt{1 - s^2}}, \qquad \mathbf{E}s^{T_b} = (\mathbf{E}s^{T_1})^b.$$

  (Hint: $\cosh^{-1}(z) = \log(z + \sqrt{z+1}\sqrt{z-1})$ for any $z > 1$.)
- If $0 < p < 1/2$, show that $\mathbf{P}(T_b < \infty) = e^{-\lambda b}$ where $\lambda := \log((1-p)/p) > 0$.
- If $0 < p < 1/2$, show that $Z := 1 + \max_{n\geq 0} Z_n$ is a geometric random variable with success probability $1 - e^{-\lambda}$.

Solution. Let $\mathcal{F}_n := \sigma(Y_0, \ldots, Y_n)$ for all $n \geq 0$. Then

$\mathbf{E}(X_{n+1} - X_n|\mathcal{F}_n) = M(\alpha)^{-n}\mathbf{E}(e^{\alpha(Y_{n+1}+Z_n)-\log M(\alpha)} - e^{\alpha Z_n}|\mathcal{F}_n)$

$= e^{\alpha Z_n}M(\alpha)^{-n}\mathbf{E}(e^{\alpha Y_{n+1}-\log M(\alpha)} - 1|\mathcal{F}_n)$, since Z_n is $\mathcal{F}_n$-measurable, and by Prop. 5.16

$= X_n\mathbf{E}(e^{\alpha Y_{n+1}-\log M(\alpha)} - 1)$, since Y_{n+1} is independent of $\mathcal{F}_n$, and by Exercise 5.11

$= X_n\Big(\dfrac{M(\alpha)}{M(\alpha)} - 1\Big) = 0.$

So, $X_1, X_2, \ldots$ is a martingale (each random variable is bounded, so $\mathbf{E}\,|X_n| < \infty$ for all $n \geq 1$, and X_n is $\mathcal{F}_n$-measurable by definition of X_n.)

Fix $n \geq 0$. If $\alpha > 0$, note that $\left|e^{\alpha Z_{n\wedge T_b}}\right| \leq e^{\alpha b}$ for all $n \geq 0$. Also, if $1/2 \leq p < 1$, then

$$M(\alpha) = pe^{\alpha} + (1-p)e^{-\alpha} \geq (1/2)(e^{\alpha} + e^{-\alpha}) > 1, \quad \text{since } \alpha > 0.$$

So, $|X_{n\wedge T_b}| \leq e^{\alpha b}$ for all $n \geq 1$. So by e.g Doob's Optional Stopping Theorem, since $|X_{0\wedge T_b}|, |X_{1\wedge T_b}|, \ldots$ are uniformly bounded hence uniformly integrable,

$$1 = \mathbf{E}X_0 = \mathbf{E}X_{T_b} = \mathbf{E}e^{\alpha X_{T_b}} M(\alpha)^{-T_b} = e^{\alpha b}\mathbf{E}M(\alpha)^{-T_b}.$$

(In the case $T_b = \infty$, the integrand is zero; actually $\mathbf{P}(T_b = \infty) = 0$ but we do not need this.)

Now, note that the function $M(\alpha)$ satisfies $M(0) = 1$ and if $1/2 \leq p < 1$, we have

$$M'(\alpha) = pe^{\alpha} + (p-1)e^{-\alpha} \geq (1/2)(e^{\alpha} + e^{-\alpha}) > 1, \quad \text{if}\, \alpha > 0.$$

So, M is a strictly increasing function of α. So if $0 < s < 1$, there exists a unique $\alpha > 0$ such that $1/s = M(\alpha)$. By the first part of the problem, we then have

$$e^{\alpha b}\mathbf{E}M(\alpha)^{-T_b} = e^{\alpha b}\mathbf{E}s^{T_b} = 1.$$

That is, $\mathbf{E}s^{T_b} = e^{-\alpha b} = (e^{-\alpha})^b = (\mathbf{E}s^{T_1})^b$.

When $p = 1/2$ we have $1/s = \cosh\alpha$, so that $\alpha = \cosh^{-1}(1/s)$ and

$$\begin{aligned}\mathbf{E}s^{T_1} = e^{-\alpha} = e^{-\cosh^{-1}(1/s)} &= \frac{1}{1/s + \sqrt{(1/s)+1}\sqrt{(1/s)-1}} \\ = \frac{s}{1 + \sqrt{s+1}\sqrt{1-s}} &= \frac{s}{1+\sqrt{1-s^2}}.\end{aligned}$$

Now, let $0 < p < 1/2$ and let $q := 1-p$. Let $0 \leq a < 0 < b$. Let $T_{a,b} = \min\{n \geq 1 \colon Z_n \in \{a,b\}\}$. Let $c := \mathbf{P}(Z_{T_{a,b}} = a)$. From Exercise 6.55 on the Homework,

$$c = \frac{1-(q/p)^b}{(q/p)^a - (q/p)^b}.$$

That is,

$$\mathbf{P}(Z_{T_{a,b}} = b) = 1 - c = \frac{(q/p)^a - 1}{(q/p)^a - (q/p)^b}.$$

Letting $a \to -\infty$, and using $0 < p < 1/2$ so that $1/2 < q < 1$ and $1 < (q/p)$,

$$\lim_{a\to-\infty} \mathbf{P}(Z_{T_{a,b}} = b) = (p/q)^b = e^{b\log(p/q)} = e^{-b\log(q/p)}.$$

We claim that

$$\mathbf{P}(T_b < \infty) = \lim_{a\to-\infty} \mathbf{P}(Z_{T_{a,b}} = b). \qquad (*)$$

Indeed, by (a modification of) Example 4.22 in the notes, $\mathbf{P}(T_{a,b} = \infty) = 0$, so for any $a < 0$,

$$\{T_b < \infty\} \supseteq \{Z_{T_{a,b}} = b\}.$$

Also, $\{Z_{T_{a,b}} = b\} \subseteq \{Z_{T_{a-1,b}} = b\} \subseteq \{Z_{T_{a-2,b}} = b\} \subseteq \cdots$, by definition of $T_{a,b}$. And

$$\cup_{a=-1}^{-\infty}\{Z_{T_{a,b}} = b\} = \{T_b < \infty\}.$$

So, $(*)$ follows by continuity of the probability law.

Finally, if $b > 0$ is a positive integer, note that

$$\{Z > b\} = \{1 + \max_{n\geq 0} Z_n > b\} = \{\max_{n\geq 0} Z_n \geq b\} = \{T_b < \infty\}.$$

So, $\mathbf{P}(Z > b) = e^{-\lambda b}$ by the third part of the problem. That is

$$\mathbf{P}(Z = b) = \mathbf{P}(Z > b - 1) - \mathbf{P}(Z > b) = e^{-\lambda(b-1)} - e^{-\lambda b} = e^{-\lambda b}(1 - e^{-\lambda}).$$