Graduate Probability, 60850, Spring 2018, Notre Dame Instructor: Steven Heilman

Name: ________________ ND ID: ____________ Date: __________

Signature: ________________.
(By signing here, I certify that I have taken this test while refraining from cheating.)

Final Exam

This exam contains 6 pages (including this cover page) and 4 problems. Check to see if any pages are missing. Enter all requested information on the top of this page.

You may *not* use the internet on this exam. You *can* use the course textbook, course notes and homeworks. You are required to show your work on each problem on the exam. The following rules apply:

- This exam is due 96 hours from now, to be submitted electronically to the email: `sheilman@nd.edu`. For every ten minutes that the exam is late, 1 point will be deducted from the score, rounded arbitrarily.
- **If you use a theorem or proposition from class/notes/book/homework you must indicate this** and explain why the theorem may be applied. It is okay to just say, "by some theorem/proposition from class."
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.

Problem	Points	Score
1	30	
2	35	
3	40	
4	45	
Total:	150	

Do not write in the table to the right. Good luck![a]

1. (30 points) This problem proves a monotone convergence theorem for conditional expectation.

   Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space. Let $0 \leq X_1 \leq X_2 \leq \cdots$ be $\mathcal{F}$-measurable random variables on $(\Omega, \mathcal{F}, \mathbf{P})$. Let $X := \lim_{n\to\infty} X_n$ be the pointwise limit of $X_1, X_2, \ldots$. Assume $\mathbf{E}X < \infty$. Let $\mathcal{G} \subseteq \mathcal{F}$ be a σ-algebra. Show that

$$\lim_{n\to\infty} \mathbf{E}(X_n|\mathcal{G}) = \mathbf{E}(X|\mathcal{G}).$$

   (Hint: first prove that the sequence $\mathbf{E}(X_1|\mathcal{G}), \mathbf{E}(X_2|\mathcal{G}), \ldots$ has a pointwise limit, almost surely.)

2. (35 points) Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space. As usual, if $X\colon \Omega \to \mathbf{R}$ is measurable, define $||X||_1 := \mathbf{E}\,|X|$. Let H be an L_1 bounded set of random variables on $(\Omega, \mathcal{F}, \mathbf{P})$, i.e. assume that $\sup_{X\in H} ||X||_1 < \infty$.

   Suppose H is not uniformly integrable. Show that there exists $\varepsilon > 0$ and there exist **disjoint** sets $A_1, A_2, \ldots \subseteq \Omega$ such that, for all $n \geq 1$, we have

$$\sup_{X\in H} \mathbf{E}\,|X|\,1_{A_n} \geq \varepsilon.$$

   (You are not allowed to use without proof Theorem 6.60 cited in the Additional Comments section of the notes.)

   (Hint: start by using Exercise 6.41 in the notes to get $\varepsilon > 0$. Let $\{\delta_{jk}\}_{1\leq j<k<\infty}$ be nonnegative real numbers such that $\sum_{1\leq j<k<\infty} \delta_{jk} < \varepsilon/2$. Next, by inducting on k, find $X_1, X_2, \ldots \in H$ and sets $B_1, B_2, \ldots \in \mathcal{F}$ such that

$$\mathbf{E}\,|X_k|\,1_{B_k} > \varepsilon \qquad \forall\ k \geq 1.$$

$$\mathbf{E}\,|X_j|\,1_{B_k} < \delta_{jk} \qquad \forall\ 1 \leq j < k.)$$

3. (40 points) Let $Y_1, Y_2, \ldots$ be independent random variables such that $\mathbf{P}(Y_n = 1) = \mathbf{P}(Y_n = -1) = 1/2 \ \forall\ n \geq 1$. Let $Y_0 := 0$. Let $Z_n := Y_0 + \cdots + Y_n$ for any $n \geq 0$. From the homework, one might wonder where the martingale $Z_0^2 - 0, Z_1^2 - 1, \ldots$ came from, and if more like it exist. In this exercise, we compute an infinite family of such martingales.

   For any $\alpha \in \mathbf{R}$ and $n \geq 0$, let

$$X_n := e^{\alpha Z_n - n \log\cosh(\alpha)}.$$

   Show that $X_0, X_1, \ldots$ is a martingale.

   Then, using the power series expansion of the exponential function, we have $X_n = \sum_{m=0}^{\infty} \frac{\alpha^m}{m!} M_{m,n}$ for some random variables $M_{1,1}, \ldots$, for any $\alpha \in \mathbf{R}$ and for any $n \geq 0$. Show that it follows that, for any $m \geq 0$, $M_{m,0}, M_{m,1}, \ldots$ is a martingale. For example, using $m = 2$ we get $M_{2,n} = Z_n^2 - n$ for all $n \geq 0$. And using $m = 4$, $M_{4,n} = Z_n^4 - 6nZ_n^2 + 2n + 3n^2$ for all $n \geq 0$. (You can assume that this formula holds for $M_{4,n}$.) Using the martingale $(M_{4,n})_{n\geq 0}$, compute $\mathbf{E}T^2$ when $T := \min\{n \geq 1 \colon Z_n \in \{-b, b\}\}$ and $b > 0$, $b \in \mathbf{Z}$.

4. (45 points) Let $0 < p < 1$. Let b be a positive integer. Let $Y_1, Y_2, \ldots$ be independent random variables such that $\mathbf{P}(Y_n = 1) =: p$ and $\mathbf{P}(Y_n = -1) = 1 - p =: q \;\forall\, n \geq 1$. Let $Y_0 := 0$. Let $Z_n = Y_0 + \cdots + Y_n$, $\forall\, n \geq 0$. Let $T_b := \min\{n \geq 1 \colon Z_n = b\}$. For any $\alpha \in \mathbf{R}$ let $M(\alpha) := \mathbf{E}e^{\alpha Y_1}$. For any $n \geq 0$, let

$$X_n := e^{\alpha Z_n}(M(\alpha))^{-n}.$$

- If $1/2 \leq p < 1$, show that $e^{\alpha b}\mathbf{E}M(\alpha)^{-T_b} = 1$ for all $\alpha > 0$.
- If $p = 1/2$ and $0 < s < 1$, show that

$$\mathbf{E}s^{T_1} = \frac{s}{1 + \sqrt{1 - s^2}}, \qquad \mathbf{E}s^{T_b} = (\mathbf{E}s^{T_1})^b.$$

  (Hint: $\cosh^{-1}(z) = \log(z + \sqrt{z+1}\sqrt{z-1})$ for any $z > 1$.)
- If $0 < p < 1/2$, show that $\mathbf{P}(T_b < \infty) = e^{-\lambda b}$ where $\lambda := \log((1-p)/p) > 0$.
- If $0 < p < 1/2$, show that $Z := 1 + \max_{n \geq 0} Z_n$ is a geometric random variable with success probability $1 - e^{-\lambda}$.

(Scratch paper)