Please provide complete and well-written solutions to the following exercises.

Due December 5, 6PM, to be submitted in brightspace, under the Content tab (so it actually is submitted to gradescope).

Homework 7

Exercise 1. Show that the Markov Chain with state space A defined in our discussion of the hard core model is actually a Markov chain that is irreducible and aperiodic.

Exercise 2. Show that the Gibbs Sampling Algorithm for a probability distribution π on Ω creates an aperiodic Markov Chain and π is reversible with respect to this Markov Chain. Conclude that π is a stationary distribution for the Markov Chain.

If it occurs that the Markov Chain is irreducible, conclude that π is the unique stationary distribution, so that the Gibbs Sampling Algorithm is a Markov Chain Monte Carlo Algorithm for simulating π.

Exercise 3. Let $G = (V, E)$ be a finite graph. Let $\lambda > 0$. The **generalized hard core model** μ_λ is a probability measure on the set $\{0,1\}^V$. Let A denote the set of all elements of $\xi \in \{0,1\}^V$ such that, if $(v, w) \in E$ then $\xi(v), \xi(w)$ are not both equal to 1. Then μ is defined so that $\mu(\xi) := 0$ for any $\xi \notin A$, and

$$\mu(\xi) := \frac{\lambda^{\sum_{v\in V}\xi(v)}}{z}, \qquad z := \sum_{\xi\in A}\lambda^{\sum_{v\in V}\xi(v)}.$$

- Show that, if $\xi \in A$, and if $v \in V$ is fixed with $\xi(w) = 0$ for all neighboring vertices w of v, then the probability that $\xi(v) = 1$ is $\lambda/(\lambda + 1)$.
- Construct an MCMC algorithm for the generalized hard cord model.

Exercise 4. Show that the transition matrix P constructed by the general case of the Metropolis Algorithm is reversible. Conclude that π is a stationary distribution for P.

Exercise 5. Show that, for an HMM, for all $x_0, \ldots, x_n \in \Omega$ and for all $y_0, \ldots, y_n \in T$,

$$\mathbf{P}(Y_n = y_n \,|\, X_0 = x_0, \ldots, X_n = x_n, Y_0 = y_0, \ldots, Y_{n-1} = y_{n-1}) = Q(x_n, y_n).$$

Exercise 6. Let $X_0, X_1, \ldots$ be a finite Markov Chain. Let $Z_0, Z_1, \ldots \in \mathbf{R}$ be discrete random variables, independent of $X_1, \ldots, X_n$ and identically distributed. Let $f\colon \Omega \times \mathbf{R} \to \mathbf{R}$. Define

$$Y_n := f(X_n, Z_n), \qquad \forall\, n \geq 0.$$

Show that $X_0, X_1, \ldots, Y_0, Y_1, \ldots$ is an HMM, and find the corresponding emission matrix Q.