Please provide complete and well-written solutions to the following exercises.

Due December 1, 9AM, to be submitted in blackboard, under the Assignments tab.

Homework 7

Exercise 1 (Birth-and-Death Chains)**.** A birth-and-death chain can model the size of some population of organisms. Fix a positive integer k. Consider the state space $\Omega = \{0, 1, 2, \ldots, k\}$. The current state is the current size of the population, and at each step the size can increase or decrease by at most 1. We define $\{(p_n, r_n, q_n)\}_{n=0}^{k}$ such that $p_n + r_n + q_n = 1$ and $p_n, r_n, q_n \geq 0$ for each $0 \leq n \leq k$, and

- $P(n, n+1) = p_n > 0$ for every $0 \leq n < k$.
- $P(n, n-1) = q_n > 0$ for every $0 < n \leq k$.
- $P(n, n) = r_n \geq 0$ for every $0 \leq n \leq k$.
- $q_0 = p_k = 0$.

Show that the birth-and-death chain is reversible.

Exercise 2. Show that the Markov Chain with state space A defined in our discussion of the hard core model is actually a Markov chain that is irreducible and aperiodic.

Exercise 3. Show that the Gibbs Sampling Algorithm for a probability distribution π on Ω creates an aperiodic Markov Chain and π is reversible with respect to this Markov Chain. Conclude that π is a stationary distribution for the Markov Chain.

If it occurs that the Markov Chain is irreducible, conclude that π is the unique stationary distribution, so that the Gibbs Sampling Algorithm is a Markov Chain Monte Carlo Algorithm for simulating π.

Exercise 4. Let $G = (V, E)$ be a finite graph. Let $\lambda > 0$. The **generalized hard core model** μ_λ is a probability measure on the set $\{0, 1\}^V$. Let A denote the set of all elements of $\xi \in \{0, 1\}^V$ such that, if $(v, w) \in E$ then $\xi(v), \xi(w)$ are not both equal to 1. Then μ is defined so that $\mu(\xi) := 0$ for any $\xi \notin A$, and

$$\mu(\xi) := \frac{\lambda^{\sum_{v \in V} \xi(v)}}{z}, \qquad z := \sum_{\xi \in A} \lambda^{\sum_{v \in V} \xi(v)}.$$

- Show that, if $\xi \in \{0, 1\}^V$, and if $v \in V$ is fixed with $\xi(w) = 0$ for all neighboring vertices w of v, then the probability that $\xi(v) = 1$ is $\lambda/(\lambda + 1)$.
- Construct an MCMC algorithm for the generalized hard cord model.

Exercise 5. Show that the transition matrix P constructed by the general case of the Metropolis Algorithm is reversible. Conclude that π is a stationary distribution for P.