Please provide complete and well-written solutions to the following exercises.

Due September 15, 9AM, to be submitted in blackboard, under the Assignments tab.

# Homework 2

**Exercise 1.** Prove the following version of the Karlin-Rubin Theorem, with the inequalities reversed in the definition of the hypotheses.

Let $\{f_\theta\}$ be a family of PDFs with the MLR property, with respect to a real-valued statistic $Y = t(X)$, where $\theta \in \Theta \subseteq \mathbf{R}$. Let $0 \leq \gamma \leq 1$. Fix $\theta_0 \in \Theta$. Consider the hypothesis $H_0 = \{\theta \geq \theta_0\}$ and the hypothesis $H_1 = \{\theta < \theta_0\}$. Let $c \in \mathbf{R}$. Consider the randomized hypothesis test $\phi\colon \mathbf{R}^n \to [0,1]$ defined by

$$\phi(x) := \begin{cases} 0 & \text{, if } t(x) > c \\ 1 & \text{, if } t(x) < c \\ \gamma & \text{, if } t(x) = c. \end{cases}$$

Define $\alpha := \mathbf{E}_{\theta_0}\phi(X)$. Let $\mathcal{T}$ be the class of all randomized hypothesis tests with significance level at most $\alpha$.

(i) $\phi$ is UMP class $\mathcal{T}$.
(iii) $\beta$, the power function of $\phi$, is nonincreasing and strictly decreasing when it takes values in $(0,1)$.

**Exercise 2.** Prove the following one-sided version of the Karlin-Rubin Theorem.

Let $\{f_\theta\}$ be a family of PDFs with the MLR property, with respect to a real-valued statistic $Y = t(X)$, where $\theta \in \Theta \subseteq \mathbf{R}$. Let $0 \leq \gamma \leq 1$. Fix $\theta_0 \in \Theta$. Consider the hypothesis $H_0 = \{\theta = \theta_0\}$ and the hypothesis $H_1 = \{\theta > \theta_0\}$. Let $c \in \mathbf{R}$. Consider the randomized hypothesis test $\phi\colon \mathbf{R}^n \to [0,1]$ defined by

$$\phi(x) := \begin{cases} 1 & \text{, if } t(x) > c \\ 0 & \text{, if } t(x) < c \\ \gamma & \text{, if } t(x) = c. \end{cases}$$

Define $\alpha := \mathbf{E}_{\theta_0}\phi(X)$. Let $\mathcal{T}$ be the class of all randomized hypothesis tests with significance level at most $\alpha$.

Then $\phi$ is UMP class $\mathcal{T}$.

**Exercise 3.** Let $X_1, \ldots, X_n$ be i.i.d. random variables. Let $X = (X_1, \ldots, X_n)$. Let $\theta > 0$. Assume that $X_1$ is uniformly distributed in the interval $[0, \theta]$. Fix $\theta_0 > 0$. Fix $0 < \alpha < 1$. Let $\mathcal{T}$ denote the set of hypothesis tests with significance level at most $\alpha$.

- Suppose we test $H_0 = \{\theta \leq \theta_0\}$ versus $H_1 = \{\theta > \theta_0\}$. Identify the set of all UMP class $\mathcal{T}$ hypothesis tests.
- Suppose we test $H_0 = \{\theta = \theta_0\}$ versus $H_1 = \{\theta \neq \theta_0\}$. Show there is a unique UMP class $\mathcal{T}$ hypothesis test in this case.

(Hint: first consider testing $\{\theta = \theta_0\}$ versus $\{\theta = \theta_1\}$ with $\theta_1 > \theta_0$, and apply the Neyman-Pearson Lemma. That is, mimic the argument of the Karlin-Rubin Theorem.) (As an aside, observe that, if you naïvely apply the Karlin-Rubin Theorem, you will not find all UMP tests, i.e. a non-strict MLR property version of the Karlin-Rubin Theorem will neglect some UMP tests.)

**Exercise 4.** Let $X_1, \ldots, X_n$ be i.i.d. random variables that are uniformly distributed in the interval $[\theta, \theta + 1]$, where $\theta \in \mathbf{R}$ is an unknown parameter. Fix $\theta_0 \in \mathbf{R}$. Suppose we want to test the hypothesis that $\theta \leq \theta_0$ versus $\theta > \theta_0$. For any $0 \leq \alpha \leq 1$, show that there exists a UMP test among tests with significance level at most $\alpha$, and this test rejects the null hypothesis when $X_{(1)} > \theta_0 + c(\alpha)$ or $X_{(n)} > \theta_0 + 1$.

On the other hand, show that the joint density of $X_1, \ldots, X_n$ does not have the MLR property with respect to any statistic (when $n > 1$). (Hint: if it did have the MLR property, what would the Karlin-Rubin Theorem imply about the UMP rejection regions?)

(Hint: for the first part, first consider testing $\{\theta = \theta_0\}$ versus $\{\theta = \theta_1\}$ with $\theta_1 > \theta_0$, and apply the Neyman-Pearson Lemma.)

**Exercise 5.** Let $\{f_\theta \colon \theta \in \mathbf{R}\}$ be a family of positive, single-variable PDFs, i.e. $f_\theta \colon \mathbf{R} \to (0, \infty)$ for all $\theta \in \mathbf{R}$. Assume that $f_\theta(x)$ is twice continuously differentiable in the parameters $\theta, x$.

Show that $\{f_\theta\}$ has the MLR property with respect to the statistic $t(x) = x$ ($x \in \mathbf{R}$) if and only if

$$\frac{\partial^2}{\partial\theta\partial x} \log f_\theta(x) \geq 0, \qquad \forall\, x, \theta \in \mathbf{R}.$$

**Exercise 6.** Suppose $X$ is a binomial distributed random variable with parameters $n = 100$ and $\theta \in [0, 1]$ where $\theta$ is unknown. Suppose we want to test the hypothesis $H_0$ that $\theta = 1/2$ versus the hypothesis $H_1$ that $\theta \neq 1/2$. Consider the hypothesis test that rejects the null hypothesis if and only if $|X - 50| > 10$.

Using e.g. the central limit theorem, do the following:

- Give an approximation to the significance level $\alpha$ of this hypothesis test
- Plot an approximation of the power function $\beta(\theta)$ as a function of $\theta$.
- (Optional) Estimate $p$ values for this test when $X = 50$, and also when $X = 70$ or $X = 90$.

**Exercise 7.** Let $X_1, \ldots, X_n$ be a real-valued random sample of size $n$ from a family of distributions $\{f_\theta \colon \theta \in \Theta\}$. Suppose $\Theta = \mathbf{R}$. Fix $\theta \in \mathbf{R}$. Denote $X := (X_1, \ldots, X_n)$. Consider a set of hypothesis tests $\phi_\alpha \colon \mathbf{R}^n \to [0, 1]$, for any $\alpha \in [0, 1]$. Assume that these tests are nested in the sense that $\phi_\alpha \leq \phi_{\alpha'}$ for all $0 \leq \alpha < \alpha' \leq 1$. Suppose we are testing the hypothesis $H_0$ that $\{\theta \leq \theta_0\}$ versus $H_1$ that $\{\theta > \theta_0\}$. Suppose also that $\{f_\theta\}$ has the monotone likelihood ratio property with respect to a statistic $Y = t(X)$ that is a continuous random variable.

- Show that the family of UMP tests with significance level at most $\alpha$ satisfies the nested property mentioned above (for all $\alpha \in [0, 1]$).
- (Optional) Show that, if $X = x$, then the $p$-value $p(x)$ satisfies

  $$p(x) = \mathbf{P}_{\theta_0}(t(X) > t(x)).$$

  (A $p$-value is defined for randomized tests as $p(X) := \inf\{\alpha \in [0, 1] \colon \phi_\alpha(X) = 1\}$.)

**Exercise 8.** We defined the MLR property so that the likelihood ratio is a strictly increasing function of the statistic. Suppose we instead defined the MLR property so that the likelihood ratio is an increasing function of the statistic. In this case, where does our proof of the Karlin-Rubin Theorem not work correctly? Explain.