Math 541B, Fall 2025, USC Instructor: Steven Heilman

Name: ____________________ USC ID: ______________ Date: ___________

Signature: ____________________. Discussion Section: ____________________

Final Exam

This exam contains 11 pages (including this cover page) and 8 problems. Enter all requested information on the top of this page.

You may *not* use your books or notes on this exam. You cannot use a calculator or any other electronic device (or internet-enabled device) on this exam. You are required to show your work on each problem on the exam. The following rules apply:

The following rules apply:

- You have 120 minutes to complete the exam.
- **If you use a theorem or proposition from class or the notes or the book you must indicate this** and explain why the theorem may be applied. It is okay to just say, "by some theorem/proposition from class."
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this. Scratch paper is at the end of the exam.

Do not write in the table to the right. Good luck![a]

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
Total:	80	

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1. (10 points) Let U be a random variable uniform on the interval $(0,1)$.

- Describe in detail a function $g\colon (0,1) \to \{1,2,3\}$ such that $g(U)$ is uniformly distributed in $\{1,2,3\}$. Prove your assertion.
- Describe in detail a function $h\colon (0,1) \to \mathbf{R}$ such that $h(U)$ has a standard Gaussian distribution (i.e. $h(U)$ has PDF given by $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$, $\forall\, x \in \mathbf{R}$.)
  Prove your assertion.

2. (10 points) Suppose $\Theta = \{\theta_0, \theta_1\}$, $\Theta_0 = \{\theta_0\}$, $\Theta_1 = \{\theta_1\}$. Let H_0 be the hypothesis $\{\theta = \theta_0\}$ and let H_1 be the hypothesis $\{\theta = \theta_1\}$. Let $\{f_{\theta_0}, f_{\theta_1}\}$ be two multivariable probability densities on $\mathbf{R}^n$. Fix $k \geq 0$. Define a **likelihood ratio test** $\phi\colon \mathbf{R}^n \to [0,1]$ to be

$$\phi(x) := \begin{cases} 1 & \text{, if } f_{\theta_1}(x) > k f_{\theta_0}(x) \\ 0 & \text{, if } f_{\theta_1}(x) < k f_{\theta_0}(x) \\ \text{(unspecified)} & \text{, if } f_{\theta_1}(x) = k f_{\theta_0}(x). \end{cases} \qquad (*)$$

Define

$$\alpha := \sup_{\theta \in \Theta_0} \beta(\theta) = \beta(\theta_0) = \mathbf{E}_{\theta_0}\phi(X). \qquad (**)$$

Let $\mathcal{T}$ be the class of all randomized hypothesis tests with significance level at most α.

Prove the following:

Any randomized hypothesis test satisfying $(*)$ is a UMP class $\mathcal{T}$ test.

3. (10 points) Suppose X is a binomial distributed random variable with parameters 2 and $\theta \in \{1/4, 3/4\}$. (That is, X is the number of heads that result from flipping two coins, where each coin has probability θ of landing heads.)
We want to test the hypothesis H_0 that $\theta = 1/4$ versus the hypothesis H_1 that $\theta = 3/4$.

   Let $\mathcal{T}$ be the set of hypothesis tests with significance level at most $1/80$.
   (Recall that the significance level of a hypothesis test $\phi\colon \mathbf{R} \to [0,1]$ is $\sup_{\theta\in\Theta_0} \mathbf{E}_\theta \phi(X)$.)

   You may take it as given that the hypothesis test

$$\phi(x) := \begin{cases} 0 & , \text{if}\, x \neq 2 \\ 1/5 & , \text{if}\, x = 2. \end{cases}$$

   is UMP class $\mathcal{T}$.

   Is ϕ unique? That is, is there another hypothesis test ψ with $\psi \neq \phi$ such that ψ is UMP class $\mathcal{T}$? Explain in detail.

4. (10 points)

   - Give an example of a Markov Chain that is not reversible. Prove your assertion.
   - Give an example of a Markov Chain where every state has period 3. Prove your assertion.

   (Let P be the transition matrix of a finite Markov chain with state space Ω. We say that the Markov chain is **reversible** if there exists a probability distribution π on Ω satisfying the following **detailed balance condition**: $\pi(x)P(x,y) = \pi(y)P(y,x)$, $\forall\, x, y \in \Omega$.)

   (For any $x \in \Omega$, let $\mathcal{N}(x) := \{n \geq 1 \colon P^n(x,x) > 0\}$. The **period** of state $x \in \Omega$ is the largest integer that divides all of the integers in $\mathcal{N}(x)$.)

5. (10 points) Let $f\colon [0,5] \to [0,3]$ be the PDF of a real-valued random variable X with maximum value $m := 3$. Let $(X_1, Y_1), (X_2, Y_2), \ldots$ be i.i.d random variables uniformly distributed in the rectangle $[0,5] \times [0,3]$.

   Using accept/reject sampling, describe how to sample a random variable Z such that the PDF of Z is f.

   Prove that the random variable Z has PDF f.

6. (10 points) Let $G = (V, E)$ be a finite graph. (So V is a finite vertex set, and $E \subseteq \{\{x, y\}\colon x, y \in V, x \neq y\}$.) Let A denote the set of all elements of $\xi \in \{0,1\}^V$ such that, if $\{v, w\} \in E$ then $\xi(v), \xi(w)$ are not both equal to 1. The **hard core model** μ is a probability measure on the set $\{0,1\}^V$ that is uniform on the set A. That is, $\mu(\xi) = 1/|A|$ for all $\xi \in A$, and $\mu(\xi) = 0$ for all $\xi \in \{0,1\}^V \setminus A$.

Define a Markov chain $X_0, X_1, \ldots$ with state space A as follows. Initialize X_0 to be the zero function on V. For any $n \geq 0$, we will define X_{n+1} using X_n. For each integer $n \geq 0$, repeat the following procedure.

- Select one $v \in V$ uniformly at random.
- Let Y_n be uniformly distributed in $\{0, 1\}$ and independent of all previously defined random variables.
- If $Y_n = 1$, and if all vertices $w \in V$ adjacent to v satisfy $X_n(w) = 0$, then set $X_{n+1}(v) := 1$. Otherwise, set $X_{n+1}(v) := 0$.
- For all $w \in V$ with $w \neq v$, define $X_{n+1}(w) := X_n(w)$.

You may take it as given that this stochastic process $X_0, X_1, \ldots$ is a Markov Chain with state space A that is irreducible and aperiodic.

Show: $X_0, X_1, \ldots$ has unique stationary distribution μ.

7. (10 points) A finite **hidden Markov model** (HMM) is a finite Markov chain $X_0, X_1, \ldots$ with state space Ω and transition matrix P, together with a stochastic process $Y_0, Y_1, \ldots$ taking values in a finite set of observed states T, and a $|\Omega| \times |T|$ emission matrix Q. The matrices P, Q are stochastic matrices (they have nonnegative entries and the sum of the entries of any row is 1.) Let μ denote the distribution of X_0. The HMM is defined such that: all $x_0, \ldots, x_n \in \Omega$ and for all $y_0, \ldots, y_n \in T$.

$$\mathbf{P}(X_0 = x_0, \ldots, X_n = x_n, Y_0 = y_0, \ldots, Y_n = y_n) = \mu(x_0) Q(x_0, y_0) \prod_{i=1}^{n} Q(x_i, y_i) P(x_{i-1}, x_i).$$

Prove: for any $y_0, \ldots, y_{n-1} \in T$, and for any $x, z \in \Omega$,

$$\mathbf{P}(X_n = x \,|\, X_{n-1} = z, Y_0 = y_0, \ldots, Y_{n-1} = y_{n-1}) = P(z, x).$$

8. (10 points) Let $X_0, X_1, \ldots, Y_0, Y_1, \ldots$ be a Hidden Markov Model.

Fix $y_0, \ldots, y_n \in T$.

For all $x \in \Omega$, define $v_0(x) := \mathbf{P}(X_0 = x, Y_0 = y_0) = \mu(x)Q(x, y_0)$.

For any $1 \leq j \leq n$, define iteratively

$$v_j(x) = \left(\max_{z \in \Omega} v_{j-1}(z)P(z,x)\right) \cdot Q(x, y_j), \qquad \forall\, x \in \Omega.$$

Define also $w_0(x) := v_0(x)$ for all $x \in \Omega$, and for any $1 \leq j \leq n$, define

$$w_j(x) := \max_{x_0,\ldots,x_{j-1} \in \Omega} \mathbf{P}(X_0 = x_0, \ldots, X_{j-1} = x_{j-1}, X_j = x, Y_0 = y_0, \ldots, Y_j = y_j).$$

Prove by induction that: for all $1 \leq j \leq n$, and for all $x \in \Omega$,

$$w_j(x) = v_j(x).$$

(Scratch paper)

(Scratch paper)