Math 541B, Fall 2023, USC Instructor: Steven Heilman

Final Exam

This exam contains 7 pages (including this cover page) and 6 problems. Enter all requested information on the top of this page.

You may use your books, notes, or a calculator on this exam. You may **not** use the internet or any internet-enabled device. You **cannot** consult with any person, other than yourself, when completing the exam.

The following rules apply:

- This exam is due at 10:00 AM PST, December 7, 2023 (72 hours to complete the exam).
- Your final submission of the exam will be a **single** pdf file, submitted in blackboard, under the Assignments tab. (No zip files, no folders, just a single .pdf file.) This file should be named LastnameFirstname.pdf . For example, if I submitted the final, the filename would be HeilmanSteven.pdf .
- Lateness will be severely penalized.
- If you do not follow the rules on this page correctly, your grade could be severely penalized.
- I am not answering any questions less than 24 hours before the due date.
- Good luck![a]

Problem	Points	Score
1	10	
2	10	
3	20	
4	10	
5	10	
6	10	
Total:	70	

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1. (10 points) Suppose X is a binomial distributed random variable with parameters 2 and $\theta \in \{1/4, 3/4\}$. (That is, X is the number of heads that result from flipping two coins, where each coin has probability θ of landing heads.)
We want to test the hypothesis H_0 that $\theta = 1/4$ versus the hypothesis H_1 that $\theta = 3/4$.

   Let $\mathcal{T}$ be the set of hypothesis tests with significance level at most 1/10.
(Recall that the significance level of a hypothesis test $\phi\colon \mathbf{R} \to [0, 1]$ is $\sup_{\theta \in \Theta_0} \mathbf{E}_\theta \phi(X)$.)

   Find a uniformly most powerful (UMP) class $\mathcal{T}$ hypothesis test $\phi\colon \mathbf{R} \to [0, 1]$.
Compute all constants that appear in the definition of ϕ. Justify your answer.

2. (10 points) Let $X_1, \ldots, X_n$ be a random sample from a Gaussian random variable with unknown mean $\mu \in \mathbf{R}$ variance one.

   Fix $\mu_0 \in \mathbf{R}$. Suppose we want to test that hypothesis H_0 that $\mu = \mu_0$ versus the alternative H_1 that $\mu \neq \mu_0$.

   - Explicitly describe the rejection region of the generalized likelihood ratio test for this hypothesis.
   - Give an explicit formula for the p-value of this hypothesis test.

3. (20 points) Let P be the transition matrix of a finite Markov chain.

All eigenvectors and eigenvalues discussed below are left eigenvectors and left eigenvalues.

- Let $\lambda \in \mathbf{C}$ be an eigenvalue of P. Show that $|\lambda| \leq 1$.
- Given an example of a transition matrix P with at least one eigenvalue λ such that λ is not a real number.
- If P is irreducible and aperiodic, show that -1 is not an eigenvalue of P.
- If P is reversible, show that all eigenvalues of P are real.
  (Hint: show that $\langle f, Pg\rangle = \langle Pf, g\rangle$, where $\langle f, g\rangle := \sum_{x\in\Omega} f(x)g(x)\pi(x)$ for all $f, g\colon \Omega \to \mathbf{R}$, where π is stationary. To define Pf, we think of f as a column vector, so the matrix P applied to f is well-defined. You can then freely use the spectral theorem for self-adjoint matrices, which implies that all eigenvalues of a self adjoint matrix are real. Also, as a hint for the next part of the problem, P has an orthonormal basis of eigenvectors.)
- Let $\gamma := 1 - \max\{|\lambda| : \lambda \text{ is an eigenvalue of } P \text{ and } \lambda \neq 1\}$. Suppose P is irreducible and reversible with stationary distribution π. Show that, for all $n \geq 1$ and for all $f : \Omega \to \mathbf{R}$, we have
  $$\mathrm{Var}_\pi(P^n f) \leq (1-\gamma)^{2n}\mathrm{Var}_\pi(f).$$
  Here $\mathrm{Var}_\pi(f) := \mathbf{E}_\pi(f - \mathbf{E}_\pi f)^2$ and $\mathbf{E}_\pi f = \sum_{x\in\Omega} f(x)\pi(x)$.
  (You don't need to show this, but this inequality leads to a bound on the mixing time of a Markov Chain in terms of γ.)

4. (10 points) Suppose you can freely sample any number of i.i.d. Gaussian random variables with mean zero and variance one (on a computer).

   Give an MCMC algorithm for estimating the integral

$$\int_{-\infty}^{\infty} e^{-x^2-x^4-x^8} e^x \frac{dx}{\alpha}, \qquad (*)$$

   where $\alpha := \int_{-\infty}^{\infty} e^{-x^2-x^4-x^8} dx$ is an **unknown** quantity. (That is, you should not need to estimate α at all, in order to estimate the integral $(*)$.)

   (Hint: Even though we didn't cover a continuous version of MCMC, all random variables on a computer are discrete, so the discrete versions of MCMC we dealt with should be sufficient to do this problem.)

   (Hint: Consider approximating the integral by a Riemann sum, or something similar to that.)

5. (10 points) Give an example of a Markov Chain that is **not** reversible.
   Prove your assertion.

6. (10 points) Give an example of a transition matrix P for a finite Markov Chain such that:

   - The Markov Chain is reversible.
   - There is a stationary distribution π that does **not** satisfy the detailed balance condition.

   Prove your assertions.