Math 541B, Fall 2023, USC Instructor: Steven Heilman

Name: ______________________ USC ID: ______________ Date: ___________

Signature: ______________________. Discussion Section: ______________________
(By signing here, I certify that I have taken this test while refraining from cheating.)

Exam 2

This exam contains 7 pages (including this cover page) and 5 problems. Enter all requested information on the top of this page.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- You have 50 minutes to complete the exam, starting at the beginning of class.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this. Scratch paper appears at the end of the document.

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total:	50	

Do not write in the table to the right. Good luck![a]

1. (10 points) Let $\beta_1, \ldots \beta_p$ be real numbers. Show that the following two conditions are equivalent.

   - $\beta_1 = \cdots = \beta_p$
   - For any $c_1, \ldots, c_p \in \mathbf{R}$ with $\sum_{i=1}^{p} c_i = 0$, we have

$$\sum_{i=1}^{p} c_i \beta_i = 0.$$

2. (10 points) Let $\mu \in \mathbf{R}$ and let $0 < \sigma < \infty$. Let $X_1, \ldots, X_n$ be i.i.d. real-valued random variables each with mean μ and variance σ^2. Let $h\colon \mathbf{R} \to \mathbf{R}$ be a function such that h' exists and is continuous. Let $\overline{X}_n := (X_1 + \cdots + X_n)/n$. Let $Y_n := h(\overline{X}_n)$.

   Show that the jackknife estimator of the variance of Y_n converges almost surely to the same estimate of the variance you get by applying the Delta Method to Y_n.

3. (10 points) Let $X_1, \ldots, X_n$ be a random sample from a Gaussian random variable with unknown mean $\mu \in \mathbf{R}$ and unknown variance $\sigma^2 > 0$.

   Fix $\mu_0 \in \mathbf{R}$. Suppose we want to test that hypothesis H_0 that $\mu = \mu_0$ versus the alternative H_1 that $\mu \neq \mu_0$.

   Explicitly describe the rejection region of the generalized likelihood ratio test for this hypothesis.

4. (10 points) Let X_1, X_2, X_3 be i.i.d. continuous random variables such that X_1 has PDF $\{f_\theta \colon \theta \in \Theta\}$. Let W_1, W_2, W_3 be a bootstrap sample from X_1, X_2, X_3. Let Y denote the sample median of X_1, X_2, X_3. (That is, Y is the middle value among X_1, X_2, X_3, which is unique with probability one since the random variables are continuous.)

   - Describe the distribution of $(W_{(1)}, W_{(2)}, W_{(3)})$.
   - Describe the bootstrap estimator of Y. (Simplify your answer to the best of your ability.)

5. (10 points) Let $X = (X_1, \ldots, X_n)$ be a random sample of size n from a family of distributions $\{f_\theta \colon \theta \in \Theta\}$. Fix $\theta_0 \in \Theta \subseteq \mathbf{R}$. Suppose we test the hypothesis H_0 that $\{\theta = \theta_0\}$ versus the alternative $\{\theta \neq \theta_0\}$. Suppose the Fisher information of X_1 exists, is finite and nonzero, and the MLE exists, is unique, and is consistent.

   Let $\lambda(X) := \frac{\sup_{\theta \in \Theta} f_\theta(X)}{\sup_{\theta \in \Theta_0} f_\theta(X)}$ denote the generalized likelihood ratio statistic.

   Give a sketch of the proof of the following: If H_0 is true, then $2 \log \lambda(X)$ converges in distribution as $n \to \infty$ to a chi-squared random variable with one degree of freedom.

   (This Theorem, which we sketched in the notes, is known as Wilks' Theorem.)

   (Hint: Perform a second order Taylor expansion of the log-likelihood $\ell(\theta)$ at the point Y where $Y = Y_n$ is the MLE of θ, and recall that $\mathbf{E}_{\theta_0} \ell''(\theta_0) = -n I_{X_1}(\theta_0)$.)

   (You are allowed in your proof sketch to ignore technicalities, e.g. you can ignore error terms in the Taylor expansion, and you can freely assume that the MLE is consistent. You can also freely use that $\sqrt{n}(Y_n - \theta_0)$ converges in distribution to a mean zero Gaussian as $n \to \infty$, with variance $1/I_{X_1}(\theta_0)$.)

(Scratch paper)