Graduate Statistics, 541A, Spring 2023, USC Instructor: Steven Heilman

Name: ______________________ USC ID: ______________ Date: ___________

Signature: ______________________.
(By signing here, I certify that I have taken this test while refraining from cheating.)

Final Exam

This exam contains 13 pages (including this cover page) and 8 problems. Check to see if any pages are missing. Enter all requested information on the top of this page.

You may *not* use your book or any calculator on this exam. You *cannot* use your homeworks. You are required to show your work on each problem on the exam. The following rules apply:

- You have 120 minutes to complete the exam.
- **If you use a theorem or proposition from class or the notes or the book you must indicate this** and explain why the theorem may be applied. It is okay to just say, "by some theorem/proposition from class."
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this. Scratch paper appears at the end of the document.

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
Total:	80	

Do not write in the table to the right. Good luck![a]

Reference sheet

Below are some definitions that may be relevant.

We say that a sequence of random variables $X_1, X_2, \ldots \colon \Omega \to \mathbf{R}$ **converges in probability** to a random variable $X \colon \Omega \to \mathbf{R}$ if: for all $\varepsilon > 0$

$$\lim_{n\to\infty} \mathbf{P}(|X_n - X| > \varepsilon) = 0.$$

We say that a sequence of real-valued random variables $X_1, X_2, \ldots$ **converges in distribution** to a real-valued random variable X if, for any $t \in \mathbf{R}$ such that $\mathbf{P}(X \leq t)$ is continuous at t,

$$\lim_{n\to\infty} \mathbf{P}(X_n \leq t) = \mathbf{P}(X \leq t).$$

We say that a sequence of random variables $X_1, X_2, \ldots \colon \Omega \to \mathbf{R}$ **converges in** L_2 to a random variable $X \colon \Omega \to \mathbf{R}$ if

$$\lim_{n\to\infty} \mathbf{E}\,|X_n - X|^2 = 0.$$

We say that a sequence of random variables $X_1, X_2, \ldots \colon \Omega \to \mathbf{R}$ **converges almost surely** to a random variable $X \colon \Omega \to \mathbf{R}$ if

$$\mathbf{P}(\lim_{n\to\infty} X_n = X) = 1.$$

Suppose $X = (X_1, \ldots, X_n)$ is a random sample of size n from a distribution f where $f \in \{f_\theta \colon \theta \in \Theta\}$ is a family of densities (such as an exponential family). Let $t \colon \mathbf{R}^n \to \mathbf{R}^k$, so that $Y := t(X_1, \ldots, X_n)$ is a statistic.

We say that Y is a **sufficient statistic** for θ if, for every $y \in \mathbf{R}^k$ and for every $\theta \in \Theta$, the conditional distribution of $(X_1, \ldots, X_n)$ given $Y = y$ (with respect to probabilities given by f_θ) does not depend on θ.

We say Y is **minimal sufficient** for θ if Y is sufficient for θ and, for every statistic $Z \colon \Omega \to \mathbf{R}^m$ that is sufficient for θ, there exists a function $r \colon \mathbf{R}^m \to \mathbf{R}^k$ such that $Y = r(Z)$.

We say Y is **complete** for $\{f_\theta \colon \theta \in \Theta\}$ if the following holds:

For any $f \colon \mathbf{R}^m \to \mathbf{R}$ such that $\mathbf{E}_\theta f(Y) = 0 \quad \forall\, \theta \in \Theta$, it holds that $f(Y) = 0$.

We say Y is **ancillary** for θ if the distribution of Y does not depend on θ.

Let $X, Y, Z \colon \Omega \to \mathbf{R}$ be discrete or continuous random variables. Let A be the range of Y. Define $g \colon A \to \mathbf{R}$ by $g(y) := \mathbf{E}(X|Y = y)$, for any $y \in A$. We then define the **conditional expectation** of X given Y, denoted $\mathbf{E}(X|Y)$, to be the random variable $g(Y)$.

We say that Y is **uniformly minimum variance unbiased (UMVU)** for $g(\theta)$ if, for any other unbiased estimator Z for $g(\theta)$, we have $\mathrm{Var}_\theta(Y) \leq \mathrm{Var}_\theta(Z)\ \forall\ \theta \in \Theta$.

1. (10 points)

   Let X be a random variable uniformly distributed in $[0, 1]$.

   (That is, X has PDF $f_X(x) = 1$ when $x \in [0, 1]$, and $f_X(x) = 0$ when $x \notin [0, 1]$.)

   Let Y be a random variable uniformly distributed in $[0, 1]$.

   Assume that X and Y are independent.

   - Compute $\mathbf{P}(X > 3/4)$.
   - Compute $\mathbf{E}X$.
   - Compute $\mathbf{P}(X + Y \leq 1/2)$.

2. (10 points)

   Let $X_1, X_2, \ldots$ be real-valued random variables. Let X be a real-valued random variable with finite second moment. Assume that

$$\lim_{n\to\infty} \mathbf{E}\left|X_n - X\right|^2 = 0.$$

   - Prove that $X_1, X_2, \ldots$ converges in probability to X.
   - Does $X_1, X_2, \ldots$ converges in distribution to X? Justify your answer.

   [This was a repeated homework question]

3. (10 points) Let $n \geq 2$ be an integer. Let $X_1, \ldots, X_n$ be a random sample from the Gaussian distribution with mean $\mu \in \mathbf{R}$ and variance $\sigma^2 > 0$. That is, X_1 has PDF $\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$, $\forall\, x \in \mathbf{R}$.

   Let $\overline{X}_n := (X_1 + \cdots + X_n)/n$, and let $S_n := \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X}_n)^2}$.

   Show: $(n-1)S_n^2/\sigma^2$ is a chi-squared distributed random variable with $n-1$ degrees of freedom.

   Hint: you can freely use the following fact:

$$nS_{n+1}^2 = (n-1)S_n^2 + \frac{n}{n+1}(X_{n+1} - \overline{X}_n)^2, \qquad \forall\, n \geq 2.$$

   You can also freely use that S_n is independent of $\overline{X}_n$.

   You can freely use that: a chi-squared random variable with k degrees of freedom has the same distribution as a sum squares of k i.i.d. standard Gaussians.

   [This was a repeated question from a previous exam.]

4. (10 points) Let $X := (X_1, \ldots, X_n)$ be a random sample of size n from a binomial distribution with parameters n and p. Here n is a positive (known) integer and $0 < p < 1$ is unknown. (That is, $X_1, \ldots, X_n$ are i.i.d. and X_1 is a binomial random variable with parameters n and p, so that $\mathbf{P}(X_1 = k) = \binom{n}{k} p^k (1-p)^{n-k}$ for all integers $0 \leq k \leq n$.)

   You can freely use that $\mathbf{E}X_1 = np$ and $\mathrm{Var}X_1 = np(1-p)$.

   - Computer the Fisher information $I_X(p)$ for any $0 < p < 1$.
     (Consider n to be fixed.)
   - Let Z be an unbiased estimator of p^2 (assume that Z is a function of $X_1, \ldots, X_n$). State the Cramér-Rao inequality for Z.
   - Let W be an unbiased estimator of $1/p$ (assume that W is a function of $X_1, \ldots, X_n$). State the Cramér-Rao inequality for W.

   [This was a repeated question from a previous exam.]

5. (10 points) Let $X_1, \ldots, X_n$ be i.i.d. random variables. Let $\theta > 0$ be an unknown parameter. Assume that X_1 is uniform on the interval $[0, \theta]$.

   Denote $X := (X_1, \ldots, X_n)$.

   - Is the Fisher Information $I_{X_1}(\theta)$ well-defined? Justify your answer. If $I_{X_1}(\theta)$ can be computed, simplify $I_{X_1}(\theta)$ to the best of your ability.
   - Is the Fisher Information $I_X(\theta)$ well-defined? Justify your answer. If $I_X(\theta)$ can be computed, simplify $I_X(\theta)$ to the best of your ability.

   [This was a modified homework question.]

6. (10 points) Let $X_1, \ldots, X_n$ be i.i.d. random variables. Let $0 < p < 1$ be an unknown parameter. Assume that

$$\mathbf{P}(X_1 = k) = (1-p)^{k-1}p,$$

for all positive integers $k \geq 1$.

You can freely use the following facts: $\mathbf{E}X_1 = 1/p$, $\mathrm{Var}(X_1) = (1-p)/p^2$

- Find a statistic Z such that: Z is a method of moments estimator of p, and Z is a function of $X_1, \ldots, X_n$. Justify your answer.
- Find a statistic W such that W is an MLE for p, and W is a function of $X_1, \ldots, X_n$. (Make sure to justify that an MLE exists. Is an MLE unique in this case?) (Here MLE refers to a Maximum Likelihood Estimator.)

[This was a modified qual question]

7. (10 points) Let $X_1, \ldots, X_n$ be i.i.d. random variables. Let $\lambda > 0$ be an unknown parameter. Assume that

$$\mathbf{P}(X_1 = k) = (e^{-\lambda})\frac{\lambda^k}{k!},$$

for all nonnegative integers $k \geq 0$.

You can freely use the following facts: $\mathbf{E}X_1 = \lambda$, $\mathrm{Var}(X_1) = \lambda$, $I_{X_1}(\lambda) = \frac{1}{\lambda}$.

You may assume that the MLE Y_n for λ exists and is unique, and it is given by:

$$Y_n = \frac{1}{n}\sum_{i=1}^{n} X_i.$$

- Is Y_n unbiased for λ?
- Is the sequence of estimators $Y_1, Y_2, \ldots$ consistent? That is, does this sequence of random variables converge in probability to λ as $n \to \infty$? Justify your answer.
- What happens to the quantity $\mathrm{Var}(Y_n)$ as $n \to \infty$? More specifically, does this quantity asymptotically achieve the lower bound in the Cramér-Rao Inequality (for unbiased estimators of λ)? Justify your answer.
- Describe all unbiased estimators of λ that achieve equality in the Cramér-Rao inequality. (Assume such an estimator is a function of $X_1, \ldots, X_n$, for fixed n.)

8. (10 points) Let $X_1, \ldots, X_n$ be i.i.d. random variables. Let $0 < p < 1$ be an unknown parameter. Assume that

$$\mathbf{P}(X_1 = k) = (1-p)^{k-1}p,$$

for all positive integers $k \geq 1$.

- Find a statistic Z such that: Z is complete for p, Z is sufficient for p, and Z is a function of $X_1, \ldots, X_n$. Justify your answer. (If you want to, you can use a result from the homework to do this part of the question.)
- Find a statistic W such that W is unbiased for p, and such that W is UMVU for p. (W should be a function of $X_1, \ldots, X_n$.) Justify your answer.

(Hint: you can use that a sum of n independent geometric random variables has the same distribution as Y where $\mathbf{P}(Y = m) = \binom{m-1}{n-1}p^n(1-p)^{m-n}$ for all integers $m \geq n$.)

[This was a modified qual question]

(Scratch paper 1)

(Scratch paper 2)

(Scratch paper 3)