541A Midterm 2 Solutions[1]

1. Question 1

Let X, Y be random variables such that (X, Y) is uniformly distributed in the region

$$\{(x, y) \in \mathbf{R}^2 \colon x^2 + y^2 \leq 1\}.$$

Compute the following quantities:

- $\mathbf{E}(X|Y)$.
- $\mathbf{E}[\mathbf{E}(X|Y)]$.

Solution. If $y \in [-1, 1]$, then

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)dx = \int_{x=-\sqrt{1-y^2}}^{x=\sqrt{1-y^2}} \frac{1}{\pi}dx = \frac{2}{\pi}\sqrt{1-y^2}.$$

Otherwise, $f_Y(y) = 0$. So, if $x^2 + y^2 \leq 1$

$$\begin{aligned}\mathbf{E}(X|Y = y) &= \int_{-\infty}^{\infty} x f_{X|Y}(x, y)dx = \int_{-\infty}^{\infty} x\frac{f_{X,Y}(x, y)}{f_Y(y)}dx \\ &= \int_{x=-\sqrt{1-y^2}}^{x=\sqrt{1-y^2}} \frac{1/\pi}{(2/\pi)\sqrt{1-y^2}}xdx = ((1-y)^2 - (y-1)^2)\frac{1}{2-y} = 0.\end{aligned}$$

And $\mathbf{E}(X|Y = y)$ is undefined when $y \notin [-1, 1]$, since $f_Y(y) = 0$ when $y \notin [-1, 1]$.

Then, by definition of $\mathbf{E}(X|Y)$, we have

$$\mathbf{E}(X|Y) = 0.$$

2. Question 2

Let $X := (X_1, \ldots, X_n)$ be a random sample of size n from a binomial distribution with parameters n and p. Here n is a positive (known) integer and $0 < p < 1$ is unknown. (That is, $X_1, \ldots, X_n$ are i.i.d. and X_1 is a binomial random variable with parameters n and p, so that $\mathbf{P}(X_1 = k) = \binom{n}{k}p^k(1-p)^{n-k}$ for all integers $0 \leq k \leq n$.)

You can freely use that $\mathbf{E}X_1 = np$ and $\mathrm{Var}X_1 = np(1-p)$.

- Computer the Fisher information $I_X(p)$ for any $0 < p < 1$.
  (Consider n to be fixed.)
- Let Z be an unbiased estimator of p (assume that Z is a function of $X_1, \ldots, X_n$). State the Cramér-Rao inequality for Z.
- Let W be an unbiased estimator of p^3 (assume that W is a function of $X_1, \ldots, X_n$). State the Cramér-Rao inequality for W.

Solution. Using that the information of independent random variables is the sum of the informations, using the alternate definition of Fisher information using the variance, and

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using that the variance is unchanged by adding a constant inside the variance,

$$
\begin{aligned}
I_X(p) = nI_{X_1}(p) &= n\mathrm{Var}_p\Big(\frac{d}{dp}\Big[\log\Big(\binom{n}{X_1}p^{X_1}(1-p)^{n-X_1}\Big)\Big]\Big) \\
&= n\mathrm{Var}_p\Big(\frac{d}{dp}\Big[\log\binom{n}{X_1} + X_1\log p + (n-X_1)\log(1-p)\Big]\Big) \\
&= n\mathrm{Var}_p\Big(\frac{d}{dp}\Big[X_1\log p + (n-X_1)\log(1-p)\Big]\Big) \\
&= n\mathrm{Var}_p\Big(\frac{1}{p}X_1 - \frac{1}{1-p}(n-X_1)\Big) = n\mathrm{Var}_p\Big(\Big[\frac{1}{p}+\frac{1}{1-p}\Big]X_1\Big) \\
&= n\Big[\frac{1}{p}+\frac{1}{1-p}\Big]^2\mathrm{Var}_pX_1 = n\Big[\frac{1}{p(1-p)}\Big]^2 np(1-p) = \frac{n^2}{p(1-p)}
\end{aligned}
$$

The Cramér-Rao inequality says, if $g(p) := \mathbf{E}_pZ$, then

$$\mathrm{Var}_p(Z) \geq \frac{|g'(p)|^2}{I_X(p)}.$$

If $g(p) = p$, then $g'(p) = 1$, so we get

$$\mathrm{Var}_p(Z) \geq \frac{1}{I_X(p)} = \frac{p(1-p)}{n^2}.$$

If $g(p) = p^3$, then $g'(p) = 3p^2$, so we get

$$\mathrm{Var}_p(Z) \geq \frac{9p^4}{I_X(p)} = 9p^4\frac{p(1-p)}{n^2}.$$

3. Question 3

Let X be a binomial random variable with parameters n and p. Here n is a positive (known) integer and $0 < p < 1$ is unknown. That is, $\mathbf{P}(X = k) = \binom{n}{k}p^k(1-p)^{n-k}$ for all integers $0 \leq k \leq n$.)

Prove that no UMVU exists for the quantity $1/p$. (The sample size in this case is one.)

Solution. No unbiased estimate exists for the quantity $1/p$. Write $Y = t(X_1)$. Then $\mathbf{E}_\theta t(X_1) = \sum_{j=0}^{n}\binom{n}{j}t(j)p^j(1-p)^{n-j}$ and this is a polynomial in p. In particular, this quantity is bounded as $p \to 0$. However, the quantity $1/p$ is unbounded as $p \to 0$. Therefore, there is no choice of t such that $\sum_{j=0}^{n}\binom{n}{j}t(j)p^j(1-p)^{n-j} = 1/p$ for all $0 < p < 1$. That is, no unbiased estimator exists for $1/p$. In particular, no UMVU exists for $1/p$.

4. Question 4

Let $n \geq 2$. Let $X_1, \ldots, X_n$ be a random sample from the Gaussian distribution with unknown mean $\mu \in \mathbf{R}$ and unknown variance $\sigma^2 > 0$.

Find the UMVU for μ^3.

(When you find the UMVU, denote it by Y_n, and you must assume that Y_n is a function of $X_1, \ldots, X_n$.)

(In this question you can freely cite facts from the homework.)

(You can freely use the following computations: $\mathbf{E}\overline{X}_n^2 = \mu^2 + \sigma^2/n$, and $\mathbf{E}\overline{X}_n^3 = \mu^3 + 3\mu\sigma^2/n]$, and $\mathbf{E}S_n^2 = \sigma^2$.)

(Recall that $\overline{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$ and $S_n := \sqrt{\frac{1}{n-1}\sum_{i=1}^n (X_i - \overline{X}_n)^2}$.)

Solution. Using the provided computations, consider

$$Y_n := \overline{X}_n^3 - 3\overline{X}_n S_n^2/n.$$

Recalling that $\overline{X}_n$ and S_n are independent, we have

$$\mathbf{E}Y_n = \mathbf{E}\overline{X}_n^3 - 3\mathbf{E}\overline{X}_n\mathbf{E}S_n^2/n = \mu^3 + 3\mu\sigma^2/n - 3\mu\sigma^2/n = \mu^3.$$

From the Factorization Theorem and an exercise from the homework, $(\overline{X}, S^2)$ is complete sufficient for (μ, σ^2). So Y_n is UMVU for μ^3 (with fixed σ) , since Y_n is a function of the complete sufficient statistic $Z = (\overline{X}, S^2)$. So, the Lehmann-Scheffé Theorem implies that $Y_n = \mathbf{E}(Y_n|Z)$ is UMVU for μ^3 (since Y_n is unbiased for μ^3.)