541A Midterm 2 Solutions[1]

1. Question 1

Let Y, Z be a statistics, and suppose Z is sufficient for $\{f_\theta : \theta \in \Theta\}$. Show that $W := \mathbf{E}_\theta(Y|Z)$ does not depend on θ. That is, there is a function $t : \mathbf{R}^n \to \mathbf{R}$ that does not depend on θ such that $W = t(X)$, where X is the random sample. (You may assume that X, Y, Z are all discrete.)

Solution. Since Z is sufficient for θ, the conditional distribution of the sample X conditioned on Z does not depend on θ. So, if $h : \mathbf{R}^n \to \mathbf{R}$ and if $Y = h(X)$, then

$$\mathbf{E}_\theta(h(X)|Z = z) = \sum_{x \in \mathbf{R}} h(x)\mathbf{P}_\theta(X = x|Z = z) =: g(z),$$

and the expression on the right does not depend on θ, by assumption. So, $\mathbf{E}_\theta(h(X)|Z) = g(Z)$, and since Z is a statistic, we can write $Z = f(X)$, so that $\mathbf{E}_\theta(h(X)|Z) = g(f(X))$, where both g, f have no dependence on θ.

2. Question 2

Let X, Y be random variables such that (X, Y) is uniformly distributed in the region

$$\{(x, y) \in \mathbf{R}^2 : y \geq 0,\, x + y \leq 1,\, -x + y \leq 1\}.$$

Compute

$$\mathbf{E}(X|Y).$$

Solution. If $y \in [0, 1]$,

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)dx = \int_{x=y-1}^{x=1-y} dx = 1 - y - (y - 1) = 2 - 2y.$$

Otherwise, $f_Y(y) = 0$. So, if $y \in [0, 1]$

$$\begin{aligned}\mathbf{E}(X|Y = y) &= \int_{-\infty}^{\infty} x f_{X|Y}(x, y)dx = \int_{-\infty}^{\infty} x\frac{f_{X,Y}(x, y)}{f_Y(y)}dx \\ &= \int_{x=y-1}^{x=1-y} \frac{1}{2 - 2y}xdx = ((1 - y)^2 - (y - 1)^2)\frac{1}{2 - 2y} = 0.\end{aligned}$$

And $\mathbf{E}(X|Y = y)$ is undefined when $y \notin [0, 1]$, since $f_Y(y) = 0$ when $y \notin [0, 1]$.

Then, by definition of $\mathbf{E}(X|Y)$, we have

$$\mathbf{E}(X|Y) = 0.$$

3. Question 3

- Let $X := (X_1, \ldots, X_n)$ be a random sample of size n from a Gaussian distribution with unknown mean $\mu \in \mathbf{R}$ and unknown variance $\sigma^2 > 0$. (That is, $X_1, \ldots, X_n$ are i.i.d. and X_1 is a Gaussian with mean μ and variance σ^2.)

  Computer the Fisher information $I_X(\sigma)$ for any $\sigma > 0$. (Consider $\mu \in \mathbf{R}$ to be fixed.)

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- Give an example of a random variable Z whose distribution depends on a parameter $\theta \in \Theta$ such that $I_Z(\theta)$ does not exist for some $\theta \in \Theta$.

Solution. Recall that the density of X_1 is

$$f_\sigma(x) := \frac{1}{\sigma\sqrt{2\pi}} \exp(-(x-\mu)^2/[2\sigma^2]), \qquad \forall\, x \in \mathbf{R}.$$

Using that the information of independent random variables is the sum of the informations, using the alternate definition of Fisher information using the variance, and using that the variance is unchanged by adding a constant inside the variance,

$$\begin{aligned} I_X(\sigma) = nI_{X_1}(\sigma) &= n\mathrm{Var}_\sigma\Big(\frac{d}{d\sigma}\Big[-\log(\sigma\sqrt{2\pi}) + \frac{-(X_1-\mu)^2}{2\sigma^2}\Big]\Big) \\ &= n\mathrm{Var}_\sigma\Big(\frac{d}{d\sigma}\frac{-(X_1-\mu)^2}{2\sigma^2}\Big) = n\sigma^{-6}\mathrm{Var}_\sigma[(X_1-\mu)^2] \\ &= n\sigma^{-6}\mathbf{E}_\sigma((X_1-\mu)^4 - \sigma^4) = 2n\sigma^{-2}. \end{aligned}$$

Here we used that the fourth moment of a standard gaussian is 3, i.e. $\mathbf{E}([X_1-\mu]/\sigma)^4 = 3$, so that $\mathbf{E}(X_1-\mu)^4 = 3\sigma^4$.

To give an example where the Fisher information does not exist, consider a random variable that is uniform in $[0,\theta]$ with $\theta > 0$ unknown. Then the region where the PDF is nonzero depends on θ, so the Fisher information is not well-defined in this case (the derivative $\frac{d}{d\theta}1_{[0,\theta]}$ is not well-defined in this setting.)

4. Question 4

Let $X_1,\ldots,X_n$ be a random sample from the Bernoulli distribution with unknown parameter $0 < p < 1$, so that, for all $1 \leq i \leq n$,

$$\mathbf{P}(X_i = 1) = p, \qquad \mathbf{P}(X_i = 0) = 1-p.$$

- Find a complete sufficient statistic for p. (As usual, justify your answer.)
- Find the UMVU for p^3. (You may assume $n \geq 3$.)
  (Hint: $X_1X_2X_3$ is an estimator for p^3.)

Solution. The joint distribution of $(X_1,\ldots,X_n)$ is

$$\begin{aligned} \mathbf{P}(X_1 = x_1,\ldots,X_n = x_n) &= \prod_{i=1}^n p^{x_i}(1-p)^{1-x_i} = p^{\sum_{i=1}^n x_i}(1-p)^{n-\sum_{i=1}^n x_i} \\ &= e^{(\log p - \log(1-p))\sum_{i=1}^n x_i + n\log(1-p)}. \end{aligned}$$

From the Factorization Theorem, $\sum_{i=1}^n X_i$ is sufficient for p. Also, by some exercises from the homework, $\sum_{i=1}^n X_i$ is complete and sufficient for the exponential family (since the set $\{(\log p - \log(1-p))\colon p \in (0,1)\}$ contains an open interval in $\mathbf{R}$). Alternatively, note that $Z := \sum_{i=1}^n X_i$ has the binomial distribution and Z is complete, as shown in Example 5.21 in the notes. In any case, $Z := \sum_{i=1}^n X_i$ is complete and sufficient for θ. Also, $\frac{1}{n}\sum_{i=1}^n X_i$ is unbiased for p, so $\frac{1}{n}\sum_{i=1}^n X_i$ is UMVU for p.

Suppose we want to estimate p^3. We use the unbiased estimate $Y := X_1X_2X_3$ (noting that $\mathbf{E}_pY = \mathbf{E}_\theta X_1 \mathbf{E}_\theta X_2 \mathbf{E} X_3 = p^3$, by independence.) The UMVU is then $\mathbf{E}(Y|Z)$. Let $3 \leq z \leq n$

be an integer. Note that $Y = 1$ when $X_1 = X_2 = X_3 = 1$ and $Y = 0$ otherwise. So,

$$\begin{aligned}
\mathbf{E}_p(Y|Z=z) &= \mathbf{E}_p(1_{X_1=X_2=X_3=1}|Z=z) = \mathbf{P}_p(X_1 = X_2 = X_3 = 1|Z=z) \\
&= \mathbf{P}_p(X_1 = X_2 = X_3 = 1|\sum_{i=1}^n X_i = z) = \frac{\mathbf{P}_p(X_1 = X_2 = X_3 = 1, \sum_{i=1}^n X_i = z)}{\mathbf{P}_p(\sum_{i=1}^n X_i = z))} \\
&= \frac{\mathbf{P}_p(X_1 = X_2 = X_3 = 1, \sum_{i=4}^n X_i = z-3)}{\mathbf{P}_p(\sum_{i=1}^n X_i = z))} = \frac{p^3 \binom{n-3}{z-3} p^{z-3}(1-\theta)^{n-z}}{\binom{n}{z} p^z (1-p)^{n-z}} \\
&= \frac{1}{n(n-1)(n-2)} \frac{(n-z)!z!}{(n-z)!(z-3)!} = \frac{z(z-1)(z-2)}{n(n-1)(n-2)}.
\end{aligned}$$

Additionally, $\mathbf{E}_p(Y|Z=z) = 0 = \frac{z(z-1)(z-2)}{n(n-1)(n-2)}$ for $z = 0, 1$ and for $z = 2$ (since Z taking such a value implies that $Y = 0$). So,

$$\mathbf{E}_p(Y|Z=z) = \frac{z(z-1)(z-2)}{n(n-1)(n-2)}, \qquad \forall\, 0 \le z \le n.$$

So, the UMVU is

$$\mathbf{E}_\theta(Y|Z) = \frac{Z(Z-1)(Z-2)}{n(n-1)(n-2)}.$$