541A Midterm 1 Solutions[1]

1. Question 1

Let $\phi\colon \mathbf{R}\to\mathbf{R}$. We say that ϕ is **convex** if, for any $y\in\mathbf{R}$, there exists a constant a and there exists a function $L\colon\mathbf{R}\to\mathbf{R}$ defined by $L(x)=a(x-y)+\phi(y)$, $x\in\mathbf{R}$, such that $L(x)\leq\phi(x)$ for all $x\in\mathbf{R}$.

Let $X\colon\Omega\to[-\infty,\infty]$ be a random variable. Let $\phi\colon\mathbf{R}\to\mathbf{R}$ be convex. Assume that $\mathbf{E}\,|X|<\infty$ and $\mathbf{E}\,|\phi(X)|<\infty$. Prove that

$$\phi(\mathbf{E}X)\leq\mathbf{E}\phi(X).$$

Solution. Choose $y:=\mathbf{E}X$. Then there exists $a\in\mathbf{R}$ such that

$$a(x-\mathbf{E}X)+\phi(\mathbf{E}X)\leq\phi(x),\qquad\forall\,x\in\mathbf{R}.$$

Taking expected values of both sides in $x=X$, we get

$$\phi(\mathbf{E}X)=\mathbf{E}[a(X-\mathbf{E}X)+\phi(\mathbf{E}X)]\leq\mathbf{E}\phi(X).$$

2. Question 2

Let $\theta\in\mathbf{R}$. Let $Y_1,Y_2,\ldots$ be random variables such that $\sqrt{n}(Y_n-\theta)$ converges in distribution to a mean zero Gaussian random variable with variance $\sigma^2>0$ as $n\to\infty$. Let $f\colon\mathbf{R}\to\mathbf{R}$. Assume that $f'(\theta)$ exists. Let $Z_1,Z_2,\ldots$ be random variables that converge to zero in probability as $n\to\infty$. Assume that for any $n\geq1$, we have

$$\sqrt{n}[f(Y_n)-f(\theta)]=f'(\theta)\sqrt{n}(Y_n-\theta)+Z_n.\qquad(*)$$

- Prove that $\sqrt{n}(f(Y_n)-f(\theta))$ converges in distribution as $n\to\infty$ to a random variable W.
- What is the mean and variance of W? What PDF does W have?
- Prove or disprove the following statement: the variance of $\sqrt{n}(f(Y_n)-f(\theta))$ converges to the variance of W as $n\to\infty$

Solution. Slutsky's Theorem (Proposition 2.36 in the notes) and $(*)$ imply that $\sqrt{n}[f(Y_n)-f(\theta)]$ converges in distribution to a mean zero Gaussian with variance $\sigma^2(f'(\theta))^2$.

The variance of $\sqrt{n}(f(Y_n)-f(\theta))$ does not necessarily converge to that of W. For example, $f'(\theta)\sqrt{n}(Y_n-\theta)$ could have bounded variance as $n\to\infty$, but $Z_1,Z_2,\ldots$ could have variance going to infinity as $n\to\infty$ (with $Z_1,Z_2,\ldots$ independent of $Y_1,Y_2,\ldots$). (For example, suppose $\mathbf{P}$ is uniform on $(0,1)$ and $Z_n=n1_{[0,1/n]}-n1_{[1/n,2/n]}$. Then $\mathbf{E}Z_n=0$, $\mathbf{E}Z_n^2=2n$, so $\mathrm{var}(Z_n)=2n\to\infty$ as $n\to\infty$, while $Z_1,Z_2,\ldots$ converges in probability to 0 as $n\to\infty$.)

3. Question 3

Let $n\geq2$ be an integer. Let $X_1,\ldots,X_n$ be a random sample from the Gaussian distribution with mean $\mu\in\mathbf{R}$ and variance $\sigma^2>0$. That is, X_1 has PDF $\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$, $\forall$ $x\in\mathbf{R}$.

Let $\overline{X}_n:=(X_1+\cdots+X_n)/n$, and let $S_n:=\sqrt{\frac{1}{n-1}\sum_{i=1}^n(X_i-\overline{X}_n)^2}$.

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Show: $(n-1)S^2/\sigma^2$ is a chi-squared distributed random variable with $n-1$ degrees of freedom.

Hint: you can freely use the following fact:

$$nS_{n+1}^2 = (n-1)S_n^2 + \frac{n}{n+1}(X_{n+1} - \overline{X}_n)^2, \qquad \forall\, n \geq 2.$$

You can also freely use that S_n is independent of $\overline{X}_n$.

Solution. We now prove the third item. Let $\overline{X}_n := \frac{1}{n}\sum_{i=1}^n X_i$ and let $S_n^2 := \frac{1}{n-1}\sum_{i=1}^n (X_i - \overline{X}_n)^2$. In the case $n = 2$, we have $S_2^2 = \frac{1}{4}(X_1 - X_2)^2 + \frac{1}{4}(X_2 - X_1)^2 = \frac{1}{2}(X_1 - X_2)^2$. From Example 1.108 in the notes $\frac{1}{\sqrt{2}}(X_1 - X_2)$ is a mean zero Gaussian random variable with variance 1. So, S_2^2 is a chi-squared distributed random variable by the definition of a chi-squared random variable with one degree of freedom. That is, the third item of this proposition holds when $n = 2$. We now induct on n, using the hint.

From the first item, S_n is independent of $\overline{X}_n$. Also, X_{n+1} is independent of S_n by Proposition 1.61 in the notes, since S_n is a function of $X_1, \ldots, X_n$, the latter being independent of X_{n+1}. In summary, S_n is independent of $(X_{n+1} - \overline{X}_n)^2$. By the inductive hypothesis, $(n-1)S_n^2$ is a chi-squared distributed random variable with $n-1$ degrees of freedom. From Example 1.108 in the notes, $X_{n+1} - \overline{X}_n$ is a Gaussian random variable with mean zero and variance $1 + 1/n$, so that $\sqrt{n/(n+1)}(X_{n+1} - \overline{X}_n)$ is a mean zero Gaussian with variance 1. The definition of a chi-squared random variable then implies that nS_{n+1} is a chi-squared random variable with n degrees of freedom, completing the inductive step.

4. Question 4

Let $X_1, \ldots, X_n$ be i.i.d. random variables, so that X_1 has PDF $f_\theta \colon \mathbf{R} \to [0, \infty)$, where $\theta = (\theta_1, \theta_2) \in \mathbf{R}^2$ is an unknown parameter.

Let Y be a statistic (so that Y is a function of $X_1, \ldots, X_n$). In all cases below, as usual, you must **justify your answer**.

- (i) Suppose Y is sufficient for θ. Is it true that Y is sufficient for θ_1?
- (ii) Suppose Y is sufficient for θ_1, and Y is sufficient for θ_2. Is it true that Y is sufficient for θ?
- (iii) Suppose Y is minimal sufficient for θ_1, and Y is minimal sufficient for θ_2. Is it true that Y is minimal sufficient for θ?

Solution. Let $X = (X_1, \ldots, X_n)$. For (i), by assumption the PDF of $X|Y = y$ does not depend on θ. In particular the PDF of $X|Y = y$ does not depend on θ_1. So, yes, Y is sufficient for θ_1.

For (ii), by assumption the PDF of $X|Y = y$ does not depend on θ_1, and the PDF of $X|Y = y$ does not depend on θ_2. Therefore, the PDF of $X|Y = y$ does not depend on θ. So, yes, Y is sufficient for θ.

For (iii), note that Y is sufficient for θ by part (ii). Now, by minimal sufficiency, if Z is sufficient for θ_1, then Y is a function of Z. Now let W be sufficient for θ. We need to show that Y is a function of W. By part (i), W is sufficient for θ_1, so Y is a function of W.

5. Question 5

Let $X_1, \ldots, X_n$ be i.i.d. random variables, so that X_1 has PDF $f_\theta \colon \mathbf{R} \to [0, \infty)$, where $\theta \in \mathbf{R}$ is an unknown parameter.

Let Y be a statistic (so that Y is a function of $X_1, \ldots, X_n$). Answer the following questions. In all cases below, as usual, you must **justify your answer**.

(i) Does a statistic Y always exist such that Y is sufficient for θ?

(ii) Does a statistic Y always exist such that Y is a minimal sufficient statistic?

(iii) Does a statistic Y always exist such that Y is complete and ancillary for θ?

Solution. For (i), note that the whole sample $Y = (X_1, \ldots, X_n)$ is always sufficient for θ.

For (ii), yes this was a theorem in the notes.

For (iii), note that a constant function is both complete and ancillary.