Math 541a, Spring 2023, USC | Instructor: Steven Heilman

Name: ______________ USC ID: ______________ Date: ______________

Signature: ______________. Discussion Section: ______________

(By signing here, I certify that I have taken this test while refraining from cheating.)

Exam 1

This exam contains 8 pages (including this cover page) and 5 problems. Enter all requested information on the top of this page.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- You have 50 minutes to complete the exam, starting at the beginning of class.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this. Scratch paper appears at the end of the document.

Do not write in the table to the right. Good luck![a]

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total:	50	

Reference sheet

Below are some definitions that may be relevant.

We say that a sequence of random variables $X_1, X_2, \ldots \colon \Omega \to \mathbf{R}$ **converges in probability** to a random variable $X \colon \Omega \to \mathbf{R}$ if: for all $\varepsilon > 0$

$$\lim_{n\to\infty} \mathbf{P}(|X_n - X| > \varepsilon) = 0.$$

We say that a sequence of real-valued random variables $X_1, X_2, \ldots$ **converges in distribution** to a real-valued random variable X if, for any $t \in \mathbf{R}$ such that $\mathbf{P}(X \leq t)$ is continuous at t,

$$\lim_{n\to\infty} \mathbf{P}(X_n \leq t) = \mathbf{P}(X \leq t).$$

We say that a sequence of random variables $X_1, X_2, \ldots \colon \Omega \to \mathbf{R}$ **converges in** L_2 to a random variable $X \colon \Omega \to \mathbf{R}$ if

$$\lim_{n\to\infty} \mathbf{E}\, |X_n - X|^2 = 0.$$

We say that a sequence of random variables $X_1, X_2, \ldots \colon \Omega \to \mathbf{R}$ **converges almost surely** to a random variable $X \colon \Omega \to \mathbf{R}$ if

$$\mathbf{P}(\lim_{n\to\infty} X_n = X) = 1.$$

Suppose $X = (X_1, \ldots, X_n)$ is a random sample of size n from a distribution f where $f \in \{f_\theta \colon \theta \in \Theta\}$ is a family of densities (such as an exponential family). Let $t \colon \mathbf{R}^n \to \mathbf{R}^k$, so that $Y := t(X_1, \ldots, X_n)$ is a statistic.

We say that Y is a **sufficient statistic** for θ if, for every $y \in \mathbf{R}^k$ and for every $\theta \in \Theta$, the conditional distribution of $(X_1, \ldots, X_n)$ given $Y = y$ (with respect to probabilities given by f_θ) does not depend on θ.

We say Y is **minimal sufficient** for θ if Y is sufficient for θ and, for every statistic $Z \colon \Omega \to \mathbf{R}^m$ that is sufficient for θ, there exists a function $r \colon \mathbf{R}^m \to \mathbf{R}^k$ such that $Y = r(Z)$.

We say Y is **complete** for $\{f_\theta \colon \theta \in \Theta\}$ if the following holds:

For any $f \colon \mathbf{R}^m \to \mathbf{R}$ such that $\mathbf{E}_\theta f(Y) = 0 \quad \forall\, \theta \in \Theta$, it holds that $f(Y) = 0$.

We say Y is **ancillary** for θ if the distribution of Y does not depend on θ.

We say W is a **chi-squared** random variable with $p \geq 1$ degrees of freedom if W has the same distribution as $Z_1^2 + \cdots + Z_p^2$ where $Z_1, \ldots, Z_p$ are independent standard Gaussian random variables, i.e $\mathbf{P}(Z_i \leq t) = \int_{-\infty}^{t} e^{-x^2/2} dx/\sqrt{2\pi}$ for all $t \in \mathbf{R}$, for all $1 \leq i \leq p$.

1. (10 points) Let $\phi\colon \mathbf{R} \to \mathbf{R}$. We say that ϕ is **convex** if, for any $y \in \mathbf{R}$, there exists a constant a and there exists a function $L\colon \mathbf{R} \to \mathbf{R}$ defined by $L(x) = a(x-y) + \phi(y)$, $x \in \mathbf{R}$, such that $L(x) \leq \phi(x)$ for all $x \in \mathbf{R}$.

   Let $X\colon \Omega \to [-\infty, \infty]$ be a random variable. Let $\phi\colon \mathbf{R} \to \mathbf{R}$ be convex. Assume that $\mathbf{E}\,|X| < \infty$ and $\mathbf{E}\,|\phi(X)| < \infty$. Prove that

$$\phi(\mathbf{E}X) \leq \mathbf{E}\phi(X).$$

2. (10 points) Let $\theta \in \mathbf{R}$. Let $Y_1, Y_2, \ldots$ be random variables such that $\sqrt{n}(Y_n - \theta)$ converges in distribution to a mean zero Gaussian random variable with variance $\sigma^2 > 0$ as $n \to \infty$. Let $f\colon \mathbf{R} \to \mathbf{R}$. Assume that $f'(\theta)$ exists. Let $Z_1, Z_2, \ldots$ be random variables that converge to zero in probability as $n \to \infty$. Assume that for any $n \geq 1$, we have

$$\sqrt{n}[f(Y_n) - f(\theta)] = f'(\theta)\sqrt{n}(Y_n - \theta) + Z_n. \qquad (*)$$

- Prove that $\sqrt{n}(f(Y_n) - f(\theta))$ converges in distribution as $n \to \infty$ to a random variable W.
- What is the mean and variance of W? What PDF does W have?
- Prove or disprove the following statement: the variance of $\sqrt{n}(f(Y_n) - f(\theta))$ converges to the variance of W as $n \to \infty$

3. (10 points) Let $n \geq 2$ be an integer. Let $X_1, \ldots, X_n$ be a random sample from the Gaussian distribution with mean $\mu \in \mathbf{R}$ and variance $\sigma^2 > 0$. That is, X_1 has PDF $\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$, $\forall\, x \in \mathbf{R}$.

   Let $\overline{X}_n := (X_1 + \cdots + X_n)/n$, and let $S_n := \sqrt{\frac{1}{n-1} \sum_{i=1}^n (X_i - \overline{X}_n)^2}$.

   Show: $(n-1)S_n^2/\sigma^2$ is a chi-squared distributed random variable with $n-1$ degrees of freedom.

   Hint: you can freely use the following fact:

$$nS_{n+1}^2 = (n-1)S_n^2 + \frac{n}{n+1}(X_{n+1} - \overline{X}_n)^2, \qquad \forall\, n \geq 2.$$

   You can also freely use that S_n is independent of $\overline{X}_n$.

4. (10 points) Let $X_1, \ldots, X_n$ be i.i.d. random variables, so that X_1 has PDF $f_\theta \colon \mathbf{R} \to [0, \infty)$, where $\theta = (\theta_1, \theta_2) \in \mathbf{R}^2$ is an unknown parameter.

   Let Y be a statistic (so that Y is a function of $X_1, \ldots, X_n$). In all cases below, as usual, you must **justify your answer**.

   (i) Suppose Y is sufficient for θ. Is it true that Y is sufficient for θ_1?

   (ii) Suppose Y is sufficient for θ_1, and Y is sufficient for θ_2. Is it true that Y is sufficient for θ?

   (iii) Suppose Y is minimal sufficient for θ_1, and Y is minimal sufficient for θ_2. Is it true that Y is minimal sufficient for θ?

5. (10 points) Let $X_1, \ldots, X_n$ be i.i.d. random variables, so that X_1 has PDF $f_\theta\colon \mathbf{R} \to [0, \infty)$, where $\theta \in \mathbf{R}$ is an unknown parameter.

Let Y be a statistic (so that Y is a function of $X_1, \ldots, X_n$). Answer the following questions. In all cases below, as usual, you must **justify your answer**.

(i) Does a statistic Y always exist such that Y is sufficient for θ?

(ii) Does a statistic Y always exist such that Y is a minimal sufficient statistic?

(iii) Does a statistic Y always exist such that Y is complete and ancillary for θ?

(Scratch paper)