Quiz 4 occurs October 21, in the discussion section. The quiz will be based upon the problems below.

Quiz 4 Problems

Exercise 1. Let $X_1, \ldots, X_n$ be a random sample of size $n = 2$, so that X_1 is a sample from exponential distribution with unknown parameter $\theta > 0$, so that X_1 has density $\theta e^{-x\theta} 1_{x>0}$.

Suppose we want to estimate the mean

$$g(\theta) := 1/\theta.$$

- Find the UMVU for $g(\theta)$. (Hint: what condition of equality is there for the Cramér-Rao inequality?)
- Show that $\sqrt{X_1 X_2}$ has smaller mean squared error than the UMVU. That is,
$$\mathbf{E}(\sqrt{X_1 X_2} - 1/\theta)^2$$
is less than the variance of the UMVU.
- Does finding an estimator with smaller mean squared error contradict the definition of UMVU? Explain.
- (Optional) Find an estimator with even smaller mean squared error than $\sqrt{X_1 X_2}$, for all $\theta \in \Theta$.

Exercise 2. Let $f\colon \mathbf{R}^n \to \mathbf{R}$ be a convex function. Let $x \in \mathbf{R}^n$ be a local minimum of f. Show that x is in fact a global minimum of f.

Show also that if f is strictly convex, then there is at most one global minimum of f.

Now suppose additionally that f is a C^1 function (all derivatives of f exist and are continuous), and $x \in \mathbf{R}^n$ satisfies $\nabla f(x) = 0$. Show that x is a global minimum of f.

Exercise 3. Let A be a real $m \times n$ matrix. Let $x \in \mathbf{R}^n$ and let $b \in \mathbf{R}^m$. Show that the function $f\colon \mathbf{R}^n \to \mathbf{R}$ defined by $f(x) = \frac{1}{2} \|Ax - b\|^2$ is convex. Moreover, show that

$$\nabla f(x) = A^T(Ax - b), \qquad D^2 f(x) = A^T A.$$

(Here $D^2 f$ denotes the matrix of second derivatives of f.)

So, if $\nabla f(x) = 0$, i.e. if $A^T A x = A^T b$, then x is the global minimum of f. And if A has full rank, then $A^T A$ is invertible, so that $x = (A^T A)^{-1} A^T b$ is the global minimum of f.

Exercise 4. Let $f_1, \ldots, f_n\colon \mathbf{R} \to \mathbf{R}$ be n strictly convex functions on $\mathbf{R}$. Define $g\colon \mathbf{R}^n \to \mathbf{R}$ by

$$g(x_1, \ldots, x_n) := \sum_{i=1}^{n} f(x_i), \qquad \forall\, (x_1, \ldots, x_n) \in \mathbf{R}^n.$$

Show that $g\colon \mathbf{R}^n \to \mathbf{R}$ is strictly convex.

Exercise 5. Let $f\colon \mathbf{R}^2 \to \mathbf{R}$ be a C^1 function (all derivatives of f exist and are continuous). Suppose there exists $z \in \mathbf{R}$ such that, for any $x_1 \in \mathbf{R}$, we have

$$f(x_1, z) < f(x_1, x_2), \qquad \forall\, x_2 \neq z.$$

Assume also that the function

$$x_1 \mapsto f(x_1, z)$$

is strictly convex. Show that f has at most one global minimum.

Exercise 6. Let $X_1, \ldots, X_n$ be a random sample of size n from a Poisson distribution with unknown parameter $\lambda > 0$. (So, $\mathbf{P}(X_1 = k) = e^{-\lambda}\lambda^k/k!$ for all integers $k \geq 0$.)

- Find an MLE for λ.
- Find an MLE for $e^{-\lambda}$.
- How do your results compare to the previous homework, where we found two different estimators for $e^{-\lambda}$ (one from the method of moments, and the other by applying the Rao-Blackwell Theorem.)

Exercise 7. Let $X_1, \ldots, X_n$ be a random sample of size n from a Gamma distribution with unknown $\alpha > 0$ and known $\beta > 0$

- Try to find an MLE of α. (You might run into a difficulty in getting an explicit expression for α.)
- Using a computer, after fixing some possible values of $X_1, \ldots, X_n$, find an MLE of α using any computational optimization method you want to use. Can you guarantee that you have found the global maximum?