Quiz 3 occurs October 5, in the discussion section. The quiz will be based upon the problems below.

Quiz 3 Problems

Exercise 1. Let $X_1, \ldots, X_n$ be a random sample of size n from the uniform distribution on $[0, \theta]$ where $\theta > 0$ is unknown.

Show that

$$X_{(n)} = \max_{1 \leq i \leq n} X_i$$

is a sufficient statistic for θ. (From the notes, recall that $(1 + 1/n)X_{(n)}$ is consistent and unbiased for θ.)

Then, show that the sample mean is not a sufficient statistic for θ.

Exercise 2. Let $X_1, \ldots, X_n$ be a random sample of size n from the gamma distribution with $\alpha > 0$ unknown and with $\beta > 0$ known.

- Show that
$$\sum_{i=1}^{n} \log(X_i)$$
is a sufficient statistic for α. (Here log denotes the natural logarithm.)
- Show also that $\prod_{i=1}^{n} X_i$ is sufficient.
- Compute the expected value of $\prod_{i=1}^{n} X_i$ (You can freely use that $\mathbf{E}X_1 = \alpha\beta$.)
- Using a function of $\prod_{i=1}^{n} X_i$, create an estimator of α that is unbiased (when $\beta = 1$). (Hint: Rao-Blackwell. If you do use Rao-Blackwell, you do not have to simplify the conditional expectation.)

Exercise 3. Let $X_1, \ldots, X_n$ be a random sample of size n from the uniform distribution on $[\theta - 1/2, \theta + 1/2]$ where $\theta \in \mathbf{R}$ is unknown.

Show that

$$(X_{(1)}, X_{(n)})$$

is a sufficient statistic for θ.

Explain in words why $X_{(n)}$ alone, or $X_{(1)}$ alone, should not be sufficient for θ.

Exercise 4. Let Y, Z be a statistics, and suppose Z is sufficient for $\{f_\theta \colon \theta \in \Theta\}$. Show that $W := \mathbf{E}_\theta(Y|Z)$ does not depend on θ. That is, there is a function $t \colon \mathbf{R}^n \to \mathbf{R}$ that does not depend on θ such that $W = t(X)$, where X is the sample distribution.