

Quiz 1 occurs September 16, in the discussion section. The quiz will be based upon the problems below.

Quiz 2 Problems

Exercise 1. Let $X: \Omega \rightarrow \mathbf{R}$ be a random variable on a sample space Ω equipped with a probability law \mathbf{P} . For any $t \in \mathbf{R}$ let $F(t) := \mathbf{P}(X \leq t)$. For any $s \in (0, 1)$ define

$$Y(s) := \sup\{t \in \mathbf{R}: F(t) < s\}.$$

Then Y is a random variable on $(0, 1)$ with the uniform probability law on $(0, 1)$. Show that X and Y are equal in distribution. That is, $\mathbf{P}(Y \leq t) = F(t)$ for all $t \in \mathbf{R}$.

Exercise 2 (Box-Muller Algorithm). Let U_1, U_2 be independent random variables uniformly distributed in $(0, 1)$. Define

$$R := \sqrt{-2 \log U_1}, \quad \Psi := 2\pi U_2.$$

$$X := R \cos \Psi, \quad Y := R \sin \Psi.$$

Show that X, Y are independent standard Gaussian random variables. So, we can simulate any number of independent standard Gaussian random variables with this procedure.

Then, using Matlab or R (or another software package), verify that you can simulate these two Gaussian random variables by sampling from U_1, U_2 . (For example, in Matlab, you can sample from these random variables using the `rand` function.) Using about 10^7 samples of U_1, U_2 , plot a histogram of X and Y separately, and observe that it agrees with a histogram of a Gaussian.

Exercise 3. Let Y_1, Y_2, \dots be random variables such that $\sqrt{n}Y_n$ converges in distribution to a mean zero Gaussian random variable with variance $1/4$ as $n \rightarrow \infty$. Let

$$f(t) := \sin(t + 2), \quad \forall t \in \mathbf{R}$$

Show that, as $n \rightarrow \infty$, the random variables

$$\sqrt{n}(f(Y_n) - f(0))$$

converge in distribution to a random variable Z , and then compute $\mathbf{E}Z^4$.

Exercise 4. Let Y_1, Y_2, \dots be random variables such that $\sqrt{n}Y_n$ converges in distribution to a mean zero Gaussian random variable with variance $1/2$. Let

$$f(t) := \frac{e^t + e^{-t}}{2} = \cosh(t), \quad \forall t \in \mathbf{R}$$

Show that, as $n \rightarrow \infty$, the random variables

$$n(f(Y_n) - f(0))$$

converge in distribution to a random variable Z , and then compute $\mathbf{E}Z$.

Exercise 5. Recall that a gamma distributed random variable X with parameters $\alpha, \beta > 0$ satisfies

$$\mathbf{E}X = \alpha\beta, \quad \mathbf{E}X^2 = \alpha\beta^2.$$

Find the method of moments estimator for α , and also find the method of moments estimator for β .

Exercise 6. Let $\sigma > 0$, and suppose a random variable X has density

$$f(x) := \frac{1}{2\sigma} e^{-|x|/\sigma}, \quad \forall x \in \mathbf{R}.$$

Find the method of moments estimator for σ .

Exercise 7. Using a computer, create 10^4 independent samples X_1, \dots, X_n from a standard Gaussian distribution, and then observe on the computer that the sample mean and sample variance are uncorrelated. That is, plot the point $(\bar{X}, S^2 - 1)$ in the plane, and plot such a point 10^3 times (each time independent of the others, plotting all 10^3 points simultaneously). Observe that these pairs of points demonstrate zero correlation.

Exercise 8. Using a computer, create $n = 10^7$ independent samples X_1, \dots, X_n from a standard Gaussian distribution, and then create 10^7 independent samples Y_1, \dots, Y_n from a chi-squared distribution with 2 degrees of freedom, such that $Y_1, \dots, Y_n, X_1, \dots, X_n$ are all independent. Make a histogram of $X_1/\sqrt{Y_1/2}, \dots, X_n/\sqrt{Y_n/2}$. Plot also the density of Student's t -distribution with 2 degrees of freedom, to verify that the histogram agrees with this density.