Math 408, Fall 2023, USC   Instructor: Steven Heilman

Name: ____________________ USC ID: ______________ Date: ____________

Signature: ____________________. Discussion Section: ____________________
(By signing here, I certify that I have taken this test while refraining from cheating.)

Exam 2

This exam contains 8 pages (including this cover page) and 5 problems. Enter all requested information on the top of this page.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- You have 50 minutes to complete the exam, starting at the beginning of class.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this. Scratch paper appears at the end of the document.

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total:	50	

Do not write in the table to the right. Good luck![a]

Reference sheet

Below are some definitions that may be relevant.

Let $\{f_\theta \colon \theta \in \Theta\}$ be a family of multivariable probability density functions (PDFs) or probability mass functions (PMFs). Suppose $X = (X_1, \ldots, X_n)$ is a random sample of size n, so that X has distribution f_θ (i.e. f_θ is the joint distribution of $X_1, \ldots, X_n$). Let $t \colon \mathbf{R}^n \to \mathbf{R}^k$, so that $Y := t(X_1, \ldots, X_n)$ is a statistic.

We say that Y is a **sufficient statistic** for θ if, for every $y \in \mathbf{R}^k$ and for every $\theta \in \Theta$, the conditional distribution of $(X_1, \ldots, X_n)$ given $Y = y$ (with respect to probabilities given by f_θ) does not depend on θ.

Let $g \colon \Theta \to \mathbf{R}$. Let $t \colon \mathbf{R}^n \to \mathbf{R}$ and let Y be an unbiased estimator for $g(\theta)$. We say that Y is **uniformly minimum variance unbiased (UMVU)** for $g(\theta)$ if, for any other unbiased estimator Z for $g(\theta)$, we have

$$\mathrm{Var}_\theta(Y) \leq \mathrm{Var}_\theta(Z), \qquad \forall\, \theta \in \Theta.$$

Let $X, Y, Z \colon \Omega \to \mathbf{R}$ be discrete or continuous random variables. Let A be the range of Y. Define $h \colon A \to \mathbf{R}$ by $h(y) := \mathbf{E}(X|Y = y)$, for any $y \in A$. We then define the **conditional expectation** of X given Y, denoted $\mathbf{E}(X|Y)$, to be the random variable $h(Y)$.

Assume $\Theta \subseteq \mathbf{R}$. Define the **Fisher information** of X to be

$$I(\theta) = I_X(\theta) := \mathbf{E}_\theta(\frac{d}{d\theta} \log f_\theta(X))^2, \qquad \forall\, \theta \in \Theta,$$

if this quantity exists and is finite.

A **maximum likelihood estimator (MLE)** is a statistic $Y = Y_n$ taking values in Θ satisfying

$$f_Y(X) \geq f_\theta(X), \qquad \forall\, \theta \in \Theta.$$

Suppose $\Theta_0 \subseteq \Theta$ and $\Theta_1 = \Theta_0^c$. The **power** of a hypothesis test with rejection region $C \subseteq \mathbf{R}^n$ is defined to be

$$\beta(\theta) := \mathbf{P}_\theta(X \in C), \qquad \forall \theta \in \Theta.$$

The **significance level** of a hypothesis test is defined to be

$$\alpha := \sup_{\theta \in \Theta_0} \beta(\theta).$$

Here sup denotes the supremum, i.e. the least upper bound (the upper bound with the smallest value).

1. Label the following statements as TRUE or FALSE.
   If the statement is true, **EXPLAIN YOUR REASONING**.
   If the statement is false, **PROVIDE A COUNTEREXAMPLE**.

   (a) (5 points) A UMVU always exists.

   TRUE FALSE (circle one)

   (b) (5 points) An MLE always exists.

   TRUE FALSE (circle one)

2. (10 points) Let $X_1, \ldots, X_n$ be a random sample of size n from a Poisson distribution with unknown parameter $\lambda > 0$. (So, $\mathbf{P}(X_1 = k) = e^{-\lambda}\lambda^k/k!$ for all integers $k \geq 0$.)

   Find an MLE for λ. As usual, justify your answer.

   (Your MLE should be a function of $X_1, \ldots, X_n$.)

3. (10 points) Suppose X is a binomial distributed random variable with parameters 2 and $\theta \in \{1/2, 3/4\}$. (So, X has the distribution of the number of heads that appears from flipping a coin twice, where θ is the probability that a heads appears in a single coin flip.)

We want to test the hypothesis H_0 that $\theta = 1/2$ versus the hypothesis H_1 that $\theta = 3/4$.

Explicitly describe the rejection region C of the UMP (uniformly most powerful) test among all hypothesis tests with significance level at most $1/4$.

Hint: you can freely use the following facts about the PMF f_θ of X

$$\frac{f_{3/4}(0)}{f_{1/2}(0)} = \frac{1}{4}, \qquad \frac{f_{3/4}(1)}{f_{1/2}(1)} = \frac{3}{4}, \qquad \frac{f_{3/4}(2)}{f_{1/2}(2)} = \frac{9}{4}.$$

4. (10 points) Let $X := (X_1, \ldots, X_n)$ be a random sample of size n from a binomial distribution with parameters n and p. Here n is a positive (known) integer and $0 < p < 1$ is unknown. (That is, $X_1, \ldots, X_n$ are i.i.d. and X_1 is a binomial random variable with parameters n and p, so that $\mathbf{P}(X_1 = k) = \binom{n}{k} p^k (1-p)^{n-k}$ for all integers $0 \leq k \leq n$.)

   You can freely use that $\mathbf{E}X_1 = np$ and $\mathrm{Var}X_1 = np(1-p)$.

   - Computer the Fisher information $I_X(p)$ for any $0 < p < 1$.
     (Consider n to be fixed.)
   - Let Z be an unbiased estimator of p (assume that Z is a function of $X_1, \ldots, X_n$). State the Cramér-Rao inequality for Z.
   - Let W be an unbiased estimator of p^3 (assume that W is a function of $X_1, \ldots, X_n$). State the Cramér-Rao inequality for W.

5. (10 points) Let $X_1, \ldots, X_n$ be a random sample of size n from a Poisson distribution with unknown parameter $\lambda > 0$. (So, $\mathbf{P}(X_1 = k) = e^{-\lambda}\lambda^k/k!$ for all integers $k \geq 0$.)

   Let Y be the estimator $Y = 1_{\{X_1=0\}}$.

   (That is, $Y = 1$ when $X_1 = 0$, and otherwise $Y = 0$.)

   Explicitly compute $W_n := \mathbf{E}_\lambda(Y \mid \sum_{i=1}^n X_i)$.

   Simplify your answer to the best of your ability.

   (Hint: a sum of i.i.d. Poisson random variables is also Poisson.)

(Scratch paper)