408 Midterm 1 Solutions[1]

1. Question 1

TRUE/FALSE

(a) The negation of the statement
"There exists an integer j such that $j^3 - j < 5$" is:
"For every integer j, we have $j^3 - j \geq 5$."

TRUE, by the rules of negation, "There exists" is negated to "For every," and the inequality $<$ is negated to $\geq$.

(b) Let $A_1, \ldots, A_n$ be disjoint sets in a sample space Ω. Let $B \subseteq \Omega$. Then

$$\mathbf{P}(B) = \sum_{i=1}^{n} \mathbf{P}(B|A_i)\mathbf{P}(A_i).$$

FALSE. This is the total probability theorem, but with a missing assumption that $\cup_{i=1}^{n} A_i = \Omega$. A counterexample is then $B = \Omega$ and $A_1 = \cdots = A_n = \emptyset$. In this case, $\mathbf{P}(B) = 1$ but the right side is zero.

(c) Let $X_1, \ldots, X_n$ be i.i.d random variables drawn from a family of probability density functions $\{f_\theta \colon \theta \in \mathbf{R}\}$ where $f_\theta \colon \mathbf{R} \to [0, \infty)$ for all $\theta \in \mathbf{R}$. Then there must exist some integer $k \geq 1$, $\exists$ some function $t \colon \mathbf{R}^n \to \mathbf{R}^k$ and there exists some statistic $Y = t(X_1, \ldots, X_n)$ such that Y is a sufficient statistic for θ.

TRUE. the statistic $(X_1, \ldots, X_n)$ is always sufficient for θ.

(d) Let $X_1, \ldots, X_8$ be i.i.d Gaussian random variables, each with mean 1 and variance 2. Define $W := \sum_{i=1}^{8} X_i$. Then W is a Gaussian random variable with mean 1 and variance 2.

FALSE. W has mean 8 and variance 16, e.g. since the mean of W is the sum of the means of $X_1, \ldots, X_8$.

(e) Let $Y_1, Y_2, \ldots$ be a sequence of estimators such that $\mathbf{E}Y_n = 0$ for all $n \geq 1$. Then $Y_1, Y_2, \ldots$ converge in probability to 0.

FALSE. We demonstrated this with an example in class. If $\mathbf{P}$ is the uniform probability law on $[0, 1]$, and $Y_n(t) = n$ for each $0 \leq t \leq 1/2$ and $Y_n(t) = -n$ for each $1/2 < t \leq 1$, then $\mathbf{E}Y_n = 0$ for all $n \geq 1$, but Y_1, Y_2 does not converge in probability to 0, since $\mathbf{P}(|Y_n - 0| > \varepsilon) = 1$ for all $0 < \varepsilon < 1$ and for all $n \geq 1$. But convergence in probability to zero implies that $\lim_{n\to\infty} \mathbf{P}(|Y_n - 0| > \varepsilon) = 0$ for any $\varepsilon > 0$.

2. Question 2

(a) Let X be a random variable with $\mathbf{P}(X = 0) = 1/3$ and $\mathbf{P}(X = 2) = 2/3$. Compute $\mathbf{E}X$ and $\mathbf{E}(X^2)$.

By definition of X, we have $\mathbf{E}X = 0(1/3) + 2(2/3) = 4/3$ and $\mathbf{E}X^2 = 0^2(1/3) + 2^2(2/3) = 8/3$.

(b) Let Y, Z be independent random variables. Assume that $\mathbf{E}(Y^2) = 1$, $\mathbf{E}(Z^2) = 3$ and $\mathbf{E}Z = 0$. Compute $\mathbf{E}Y^4Z$ and $\mathbf{E}Y^2Z^2$.

From independence, $\mathbf{E}Y^4Z = \mathbf{E}Y^4\mathbf{E}Z = 0$, since $\mathbf{E}Z = 0$. Similarly, $\mathbf{E}Y^2Z^2 = \mathbf{E}Y^2\mathbf{E}Z^2 = 1 \cdot 3 = 3$.

(c) State the Central Limit Theorem. Make sure to include **all** assumptions.

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Let $X_1, \ldots, X_n$ be independent identically distributed random variables. Assume that $\mathbf{E}\,|X_1| < \infty$ and $0 < \mathrm{Var}(X_1) < \infty$.

Let $\mu = \mathbf{E}X_1$ and let $\sigma = \sqrt{\mathrm{Var}(X_1)}$. Then for any $-\infty \leq a \leq \infty$,

$$\lim_{n\to\infty} \mathbf{P}\left(\frac{X_1 + \cdots + X_n - \mu n}{\sigma\sqrt{n}} \leq a\right) = \int_{-\infty}^{a} e^{-t^2/2} \frac{dt}{\sqrt{2\pi}}.$$

3. Question 3

Let $Y_1, Y_2, \ldots$ be random variables such that $\sqrt{n}Y_n$ converges in distribution to a mean zero Gaussian random variable with variance 3 as $n \to \infty$. Let

$$f(t) := (t+2)^4, \qquad \forall\, t \in \mathbf{R}$$

Show that, as $n \to \infty$, the random variables

$$\sqrt{n}(f(Y_n) - f(0))$$

converge in distribution to a random variable Z, and then compute $\mathbf{E}Z^2$.

Solution. From the delta method, $\sqrt{n}(f(Y_n) - f(0))$ converges in distribution to Z where Z is a mean zero Gaussian random variable with variance $3(f'(0))^2$. Since $f'(0) = 4(2^3) = 32$, we have $\mathbf{E}Z^2 = \mathrm{var}(Z) = 3(f'(0))^2 = 3(32)^2 = 3072$.

4. Question 4

Let θ be a an unknown real parameter, and suppose a random variable X has PDF

$$f(x) := \begin{cases} \frac{1}{\theta} & \text{, if } 0 \leq x \leq \theta \\ 0 & \text{, otherwise.} \end{cases}$$

- Find a method of moments estimator for θ. Is your estimator unbiased for θ?
- Find a method of moments estimator for θ^2. Is your estimator consistent for θ^2? Justify your answer.

(In both cases, your answer should be a function of i.i.d. random variables $X_1, \ldots, X_n$, where X_1 has the same PDF as X.)

Solution. We have $\mathbf{E}X = \int_0^\theta (1/\theta)x\,dx = (1/\theta)(x^2/2)_{x=0}^{x=\theta} = \theta/2$, so that $\theta = 2\mathbf{E}X$. So, a method of moments estimator for θ is

$$\frac{2}{n}\sum_{i=1}^{n} X_i.$$

This estimator is unbiased since

$$\mathbf{E}\frac{2}{n}\sum_{i=1}^{n} X_i = \frac{2}{n}\sum_{i=1}^{n} \mathbf{E}X_i = \frac{2}{n}\sum_{i=1}^{n}(\theta/2) = \theta.$$

We also have $\mathbf{E}X^2 = \int_0^\theta (1/\theta)x^2\,dx = (1/\theta)(x^3/3)_{x=0}^{x=\theta} = \theta^2/3$, so that $\theta^2 = 3\mathbf{E}X^2$. So, a method of moments estimator for θ^2 is

$$\frac{3}{n}\sum_{i=1}^{n} X_i^2.$$

This estimator is consistent for θ^2. Since $\mathbf{E}(3X_1^2) = \theta^2$, the weak law of large numbers implies that $\frac{3}{n}\sum_{i=1}^n X_i^2$ converges in probability as $n \to \infty$ to θ^2.

5. Question 5

Let $\theta \in \mathbf{R}$ be an unknown parameter. Consider the PDF

$$f_\theta(x) := \begin{cases} e^{-(x-\theta)}, & \text{if } x \geq \theta \\ 0, & \text{if } x < \theta. \end{cases}$$

Suppose $X_1, \ldots, X_n$ is a random sample of size n, such that X_i has PDF f_θ for all $1 \leq i \leq n$.

Show that $X_{(1)} = \min_{1\leq i\leq n} X_i$ is a sufficient statistic for θ.

Solution. Let $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$. If it occurs that $\min_{1\leq i\leq n} x_i < \theta$, then some $1 \leq i \leq n$ satisfies $x_i < \theta$, so $f_\theta(x_i) = 0$ and the joint PDF $\prod_{i=1}^n f_\theta(x_i)$ is also zero. On the other hand, if it occurs that $\min_{1\leq i\leq n} x_i \geq \theta$, then all $1 \leq i \leq n$ satisfy $x_i \geq \theta$, and the joint PDF $\prod_{i=1}^n f_\theta(x_i) = \prod_{i=1}^n e^{-(x_i-\theta)}$. That is, we can write

$$\prod_{i=1}^n f_\theta(x_i) = 1_{\{\min_{1\leq i\leq n} x_i \geq \theta\}} \cdot \prod_{i=1}^n e^{-(x_i-\theta)} = e^{n\theta} 1_{\{\min_{1\leq i\leq n} x_i \geq \theta\}} \cdot \prod_{i=1}^n e^{-(x_i)}.$$

The factorization theorem then implies that $t(x) := \min_{1\leq i\leq n} x_i$ gives our sufficient statistic $Y = t(X_1, \ldots, X_n)$, since if we define $g_\theta(z) := e^{n\theta}1_{\{z\geq\theta\}}$ and $h(x) := \prod_{i=1}^n e^{-(x_i)}$, then we have written the joint PDF as

$$\prod_{i=1}^n f_\theta(x_i) = g_\theta(t(x))h(x), \qquad \forall\, x \in \mathbf{R}^n, \quad \forall\, \theta \in \mathbf{R}.$$