Math 408, Fall 2023, USC — Instructor: Steven Heilman

Name: ____________________ USC ID: ______________ Date: ____________

Signature: ____________________. Discussion Section: ____________________
(By signing here, I certify that I have taken this test while refraining from cheating.)

Exam 1

This exam contains 8 pages (including this cover page) and 5 problems. Enter all requested information on the top of this page.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- You have 50 minutes to complete the exam, starting at the beginning of class.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this. Scratch paper appears at the end of the document.

Problem	Points	Score
1	10	
2	12	
3	8	
4	10	
5	10	
Total:	50	

Do not write in the table to the right. Good luck![a]

1. Label the following statements as TRUE or FALSE. If the statement is true, **explain your reasoning**. If the statement is false, **provide a counterexample (if it is appropriate) and explain your reasoning**.

   (a) (2 points) The negation of the statement
   "There exists an integer j such that $j^3 - j < 5$" is:
   "For every integer j, we have $j^3 - j \geq 5$."

   TRUE FALSE (circle one)

   (b) (2 points) Let $A_1, \ldots, A_n$ be disjoint sets in a sample space Ω. Let $B \subseteq \Omega$. Then

   $$\mathbf{P}(B) = \sum_{i=1}^{n} \mathbf{P}(B|A_i)\mathbf{P}(A_i).$$

   TRUE FALSE (circle one)

   (c) (2 points) Let $X_1, \ldots, X_n$ be i.i.d random variables drawn from a family of probability density functions $\{f_\theta \colon \theta \in \mathbf{R}\}$ where $f_\theta \colon \mathbf{R} \to [0, \infty)$ for all $\theta \in \mathbf{R}$. Then there must exist some integer $k \geq 1$, $\exists$ some function $t \colon \mathbf{R}^n \to \mathbf{R}^k$ and there exists some statistic $Y = t(X_1, \ldots, X_n)$ such that Y is a sufficient statistic for θ.

   TRUE FALSE (circle one)

(d) (2 points) Let $X_1, \ldots, X_8$ be i.i.d Gaussian random variables, each with mean 1 and variance 2. Define $W := \sum_{i=1}^{8} X_i$. Then W is a Gaussian random variable with mean 1 and variance 2.

TRUE FALSE (circle one)

(e) (2 points) Let $Y_1, Y_2, \ldots$ be a sequence of estimators such that $\mathbf{E}Y_n = 0$ for all $n \geq 1$. Then $Y_1, Y_2, \ldots$ converge in probability to 0.

TRUE FALSE (circle one)

2. (a) (4 points) Let X be a random variable with $\mathbf{P}(X = 0) = 1/3$ and $\mathbf{P}(X = 2) = 2/3$. Compute $\mathbf{E}X$ and $\mathbf{E}(X^2)$.

Your Answer:

(b) (4 points) Let Y, Z be independent random variables.
Assume that $\mathbf{E}(Y^2) = 1$, $\mathbf{E}(Z^2) = 3$ and $\mathbf{E}Z = 0$.
Compute $\mathbf{E}Y^4Z$ and $\mathbf{E}Y^2Z^2$.

Your Answer:

(c) (4 points) State the Central Limit Theorem. Make sure to include **all** assumptions.

3. (8 points) Let $Y_1, Y_2, \ldots$ be random variables such that $\sqrt{n}Y_n$ converges in distribution to a mean zero Gaussian random variable with variance 3 as $n \to \infty$. Let

$$f(t) := (t+2)^4, \qquad \forall\, t \in \mathbf{R}$$

Show that, as $n \to \infty$, the random variables

$$\sqrt{n}(f(Y_n) - f(0))$$

converge in distribution to a random variable Z, and then compute $\mathbf{E}Z^2$.

4. (10 points) Let θ be a an unknown real parameter, and suppose a random variable X has PDF

$$f(x) := \begin{cases} \frac{1}{\theta} & \text{, if } 0 \le x \le \theta \\ 0 & \text{, otherwise.} \end{cases}$$

- Find a method of moments estimator for θ. Is your estimator unbiased for θ?
- Find a method of moments estimator for θ^2. Is your estimator consistent for θ^2? Justify your answer.

(In both cases, your estimator should be a function of i.i.d. random variables $X_1, \ldots, X_n$, where X_1 has the same PDF as X.)

5. (10 points) Let $\theta \in \mathbf{R}$ be an unknown parameter. Consider the PDF

$$f_\theta(x) := \begin{cases} e^{-(x-\theta)}, & \text{if } x \geq \theta \\ 0, & \text{if } x < \theta. \end{cases}$$

Suppose $X_1, \ldots, X_n$ is a random sample of size n, such that X_i has PDF f_θ for all $1 \leq i \leq n$.

Show that $X_{(1)} = \min_{1 \leq i \leq n} X_i$ is a sufficient statistic for θ.

(Scratch paper)