

Please provide complete and well-written solutions to the following exercises.

Due August 27, 2PM PST, to be uploaded in blackboard as a single PDF document (in the Assignments tab).

## Homework 1

**Exercise 1.** Let  $A, B, C$  be sets in a universe  $\Omega$ . Using the definitions of intersection, union and complement, prove properties (ii) and (iii) below.

- (ii)  $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ .  
 (iii)  $(A^c)^c = A$ .

(Hint: to prove property (ii), it may be helpful to first draw a Venn diagram of  $A, B, C$ . Now, let  $x \in \Omega$ . Consider where  $x$  could possibly be with respect to  $A, B, C$ . For example, we could have  $x \in A, x \notin B, x \in C$ . We could also have  $x \in A, x \in B, x \notin C$ . And so on. In total, there should be  $2^3 = 8$  possibilities for the location of  $x$ , with respect to  $A, B, C$ . Construct a [truth table](#) which considers all eight such possibilities for each side of the purported equality  $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ .)

**Exercise 2.** Let  $A_1, A_2, \dots$  be sets in some universe  $\Omega$ . Prove that  $(\bigcap_{i=1}^{\infty} A_i)^c = \bigcup_{i=1}^{\infty} A_i^c$ .

**Exercise 3.** Let  $A_1, A_2, \dots$  be sets in some universe  $\Omega$ . Let  $B \subseteq \Omega$ . Prove:

$$B \cap \left( \bigcup_{k=1}^{\infty} A_k \right) = \bigcup_{k=1}^{\infty} (A_k \cap B).$$

**Exercise 4** (Discrete Uniform Probability Law). Let  $n$  be a positive integer. Suppose we are given a finite universe  $\Omega$  with exactly  $n$  elements. Let  $A \subseteq \Omega$ . Define  $\mathbf{P}(A)$  such that  $\mathbf{P}(A)$  is the number of elements of  $A$ , divided by  $n$ . Verify that  $\mathbf{P}$  is a probability law. This probability law is referred to as the uniform probability law on  $\Omega$ , since each element of  $\Omega$  has the same probability.

**Exercise 5.** Let  $\Omega = \mathbf{R}^2$ . Let  $A \subseteq \Omega$ . Define a probability law  $\mathbf{P}$  on  $\Omega$  so that

$$\mathbf{P}(A) = \frac{1}{2\pi} \iint_A e^{-(x^2+y^2)/2} dx dy.$$

We can think of  $\mathbf{P}$  as defining the (random) position of a dart, thrown at an infinite dart board. That is, if  $A \subseteq \Omega$ , then  $\mathbf{P}(A)$  is the probability that the dart will land in the set  $A$ .

Verify that Axiom (iii) holds for  $\mathbf{P}$ . That is, verify that  $\mathbf{P}(\Omega) = 1$ . Then, compute the probability that a dart hits a circular board  $A$ , where  $A = \{(x, y) \in \mathbf{R}^2: x^2 + y^2 \leq 1\}$ .

**Exercise 6.** Let  $A, B$  be subsets of a sample space  $\Omega$ . Prove the following things:

- $A = (A \setminus B) \cup (A \cap B)$ , and  $(A \setminus B) \cap (A \cap B) = \emptyset$ .

- $A \cup B = (A \setminus B) \cup (B \setminus A) \cup (A \cap B)$ , and the three sets  $(A \setminus B)$ ,  $(B \setminus A)$ ,  $(A \cap B)$  are all disjoint. That is, any two of these sets are disjoint.

**Exercise 7.** Let  $\Omega$  be a sample space and let  $\mathbf{P}$  be a probability law on  $\Omega$ . Let  $A, B, C \subseteq \Omega$ . Prove the following things:

- $\mathbf{P}(A \cup B) \leq \mathbf{P}(A) + \mathbf{P}(B)$ .
- $\mathbf{P}(A \cup B \cup C) = \mathbf{P}(A) + \mathbf{P}(A^c \cap B) + \mathbf{P}(A^c \cap B^c \cap C)$ .

(Although the book suggests otherwise, a Venn diagram alone is not a rigorous proof. As in Exercise 1, a truth table allows us to rigorously reason about the information contained in a Venn diagram. Though, there are ways to do the problem while not directly using a truth table.)

**Exercise 8.** Let  $f: \mathbf{R} \rightarrow \mathbf{R}$  be a function. Show that

$$\cup_{y \in \mathbf{R}} \{x \in \mathbf{R}: f(x) = y\} = \mathbf{R}.$$

Also, show that the union on the left is disjoint. That is, if  $y_1 \neq y_2$  and  $y_1, y_2 \in \mathbf{R}$ , then  $\{x \in \mathbf{R}: f(x) = y_1\} \cap \{x \in \mathbf{R}: f(x) = y_2\} = \emptyset$ .