Please provide complete and well-written solutions to the following exercises.

Due June 6, in the discussion section.

Homework 8

Exercise 1. Let $t > 0$ and let $\{B(s)\}_{s\geq 0}$ be a standard Brownian motion. Compute the mean and variance of

$$\int_0^t B(s)dB(s).$$

(Hint: start with the Riemann sum, then take a limit.)

Exercise 2. Let $f\colon \mathbf{R} \to \mathbf{R}$ be a function. Let $t > 0$ and let $\{B(s)\}_{s\geq 0}$ be a standard Brownian motion. Find the distribution of

$$\int_0^t f(s)dB(s).$$

That is, find the CDF of $\int_0^t f(s)dB(s)$. (Hint: use the exercise about sums of independent Gaussians.)

Exercise 3. Using Itô's formula, write an expression for $\int_0^1 (B(s))^2 dB(s)$.

Exercise 4. Let $b > 0$. We know from calculus that $\int_0^b e^s ds = e^b - 1$.

Use $f(x) = e^x$, $x \in \mathbf{R}$, in Itô's formula to find a similar expression for $\int_0^b e^{B(s)} dB(s)$. (Note that $e^{B(s)}$ is a Geometric Brownian motion, so now we know how to take the stochastic integral of Geometric Brownian motion.)

Exercise 5 (MFE Sample Question, from an old exam)**.** Let $\{Z(t)\}_{t\geq 0}$ be a standard Brownian motion. You are given:

(i) $U(t) := 2Z(t) - 2$, for all $t \geq 0$.
(ii) $V(t) := (Z(t))^2 - t$, for all $t \geq 0$.
(iii) $W(t) := t^2 Z(t) - 2\int_0^t sZ(s)ds$, for all $t \geq 0$.

Which of the processes defined above has/have zero drift? (A stochastic process $\{U(t)\}_{t\geq 0}$ has zero drift if $dU(t) = f(Z(t), t)dZ(t)$ for some function $f\colon \mathbf{R}^2 \to \mathbf{R}$.)

Exercise 6. Let $f\colon \mathbf{R} \times [0, \infty) \to \mathbf{R}$. We write $f = f(x, t)$, where $(x, t) \in \mathbf{R} \times [0, \infty)$. Let $g\colon \mathbf{R} \to \mathbf{R}$ be a continuous and bounded function with $\int_{\mathbf{R}} |g(x)|\, dx < \infty$. We say that f satisfies the one-dimensional **heat equation** if

$$\frac{\partial f}{\partial t}(x,t) = \frac{\partial^2 f}{\partial x^2}(x,t), \quad \forall\, (x,t) \in \mathbf{R} \times [0,\infty),$$
$$f(x,0) = g(x), \qquad \forall\, x \in \mathbf{R}.$$

Show that f defined by

$$f(x,t) := \frac{1}{\sqrt{4\pi t}} \int_{\mathbf{R}} e^{-\frac{(x-y)^2}{4t}} g(y)dy = \mathbf{E}(g(B(2t)+x)), \qquad \forall\,(x,t) \in \mathbf{R} \times [0,\infty),$$

satisfies the heat equation. (Just check the first condition. You do not have to show that $\lim_{t\to 0^+} f(x,t) = g(x)$ for all $x \in \mathbf{R}$.)

Using a computer, plot the function $f(x,t)$ as a function of x for several different values of $t > 0$, using $g = 1_{[0,1]}$. Lastly, verify that $\int_{\mathbf{R}} \frac{1}{\sqrt{4\pi t}} e^{-x^2/(4t)} dx = 1$ for any $t > 0$.

Exercise 7. Let $f\colon \mathbf{R} \times [0,\infty) \to \mathbf{R}$. We write $f = f(x,t)$, where $(x,t) \in \mathbf{R} \times [0,\infty)$. Let $g\colon \mathbf{R} \to \mathbf{R}$ be a continuous and bounded function. We say that f satisfies the one-dimensional **heat equation with forcing term** $h\colon \mathbf{R} \times [0,\infty) \to \mathbf{R}$ if

$$\frac{\partial f}{\partial t}(x,t) = \frac{\partial^2 f}{\partial x^2}(x,t) + h(x,t), \quad \forall\,(x,t) \in \mathbf{R} \times [0,\infty),$$

$$f(x,0) = g(x), \qquad \forall\, x \in \mathbf{R}.$$

For any $(x,t) \in \mathbf{R} \times [0,\infty)$, define $f(x,t)$ so that

$$f(x,t) := \frac{1}{\sqrt{4\pi t}} \int_{\mathbf{R}} e^{-\frac{(x-y)^2}{4t}} g(y)dy + \int_0^t \frac{1}{\sqrt{4\pi(t-s)}} \int_{\mathbf{R}} e^{-\frac{(x-y)^2}{4(t-s)}} h(y,s)dyds.$$

Show that f satisfies the heat equation with forcing term h. (Just check the first condition.)

Exercise 8. Let $t_0 > 0$. Let $V\colon \mathbf{R} \times [0,t_0] \to \mathbf{R}$. We write $V = V(s,t)$, $s \in \mathbf{R}$, $t \in [0,t_0]$. Let $F\colon \mathbf{R} \to \mathbf{R}$. Let $r \in \mathbf{R}$, let $\sigma > 0$. We say that V satisfies the **Black-Scholes** equation if $V(s,t_0) = F(s)$ for all $s \in \mathbf{R}$, and if

$$\frac{\partial V}{\partial t} + rs\frac{\partial V}{\partial s} + \frac{\sigma^2 s^2}{2}\frac{\partial^2 V}{\partial s^2} - rV = 0.$$

Show that a solution of this equation is

$$V(s,t) := \frac{e^{-r(t_0-t)}}{\sqrt{2\pi\sigma^2(t_0-t)}} \int_0^\infty \frac{1}{z} e^{-\frac{(\log(s/z)+(r-\sigma^2/2)(t_0-t))^2}{2\sigma^2(t_0-t)}} F(z)dz.$$

(This formula should be nearly identical to the Black-Scholes Option Pricing formula from a remark in the notes, where we take $F(z) := \max(S_0 z - k, 0)$.) Instead of differentiating V directly, use the following strategy.

First, show that the Black-Scholes equation reduces to the one-dimensional heat equation

$$\frac{\partial U}{\partial \tau} = \frac{\partial^2 U}{\partial x^2},$$

where $V(s,t) = e^{ax+b\tau}U(x,\tau)$, $x = \log s$, $\tau = (\sigma^2/2)(t_0 - t)$, $a = (1/2) - r/\sigma^2$, and $b = -(1/2 + r/\sigma^2)^2$, and U satisfies the initial condition $U(x,0) = e^{-ax}F(e^x)$ for all $x \in \mathbf{R}$. (Start by differentiating V with respect to s and t, etc.) That is, the Black-Scholes equation is the heat equation, run backwards in time.

Finally, use the formula for U using Exercise 7.

Exercise 9. Let $a,b,\sigma>0$. Let $f\colon \mathbf{R}\to\mathbf{R}$ satisfying the Vasicek stochastic differential equation for any $t\in\mathbf{R}$.

$$df(t)=a(b-f(t))dt+\sigma dB(t).$$

Show that, for any $t>0$,

$$\mathbf{E}f(t)=b+e^{-at}(f(0)-b),\qquad \mathrm{var}(f(t))=\frac{\sigma^2}{2a}(1-e^{-2at}).$$

More generally, for any $s,t>0$, show that

$$\mathrm{cov}(f(t),f(u))=\mathbf{E}\big((f(t)-\mathbf{E}f(t))(f(u)-\mathbf{E}f(u))\big)=\frac{\sigma^2}{2a}(e^{-a|t-u|}-e^{-a(t+u)}).$$

Conclude that $\lim_{t\to\infty}\mathbf{E}f(t)=b$ and $\lim_{t\to\infty}\mathrm{var}(f(t))=\frac{\sigma^2}{2a}$.

Exercise 10 (Optional)**.** Using a Monte Carlo simulation, plot several sample paths of the Vasicek stochastic differential equation, with $a=b=\sigma=f(0)=1$.

Exercise 11 (Cox-Ingersoll-Ross (CIR) model)**.** Let $a,b,\sigma>0$. Let $\{B(t)\}_{t\geq 0}$ be a standard Brownian motion. The Cox-Ingersoll-Ross model models an interest rate as a (random) function $f\colon\mathbf{R}\to\mathbf{R}$ satisfying the following stochastic differential equation for any $t>0$:

$$df(t)=a(b-f(t))dt+\sqrt{f(t)}\sigma dB(t).$$

(Since f is a random function, f is also a function of the sample space, but we omit this dependence from our notation here and below.)

A priori, this stochastic differential equation is not rigorously defined, since $\sqrt{f(t)}$ will not be a real number when $f(t)<0$. In this exercise, we ignore this issue. (In actuality, if $f(0)>0$, then $f(t)<0$ occurs with probability 0.)

Unlike the Vasicek model, we might not be able to get a closed form solution of this equation. Nevertheless, we can still run a Monte Carlo simulation of this stochastic differential equation as follows. Let $f(0)=1$. Let $i,n>0$ be integers. Suppose we have inductively determined $f(i/n)$ using a Monte Carlo simulation, and we would like the determine $f((i+1)/n)$. The stochastic differential equation then suggests that

$$f((i+1)/n)\approx f(i/n)+a(b-f(i/n))(i/n)+\sqrt{f(i/n)}\sigma(B((i+1)/n)-B(i/n)).$$

This approximation is known as a finite difference scheme.

Using this approximation, plot several sample paths of the CIR model with $a=b=f(0)=\sigma=1$.

What would be the corresponding finite difference scheme for the Vasicek model?

Exercise 12 (Optional)**.** Let $\{Z(x,t)\}_{x\in\mathbf{R},t\geq 0}$ be a set of independent, standard Gaussian random variables. Suppose $f\colon\mathbf{R}\times[0,\infty)\to\mathbf{R}$ satisfies the stochastic heat equation.

$$\frac{\partial f}{\partial t}(x,t)=\frac{\partial^2 f}{\partial x^2}(x,t)+h(x,t),\quad \forall\,(x,t)\in\mathbf{R}\times[0,\infty),$$

$$f(x,0)=0,\qquad \forall\,x\in\mathbf{R}.$$

We can explicitly solve this equation by its analogy with Exercise 7. That is,

$$f(x,t) := \int_0^t \frac{1}{\sqrt{4\pi(t-s)}} \int_{\mathbf{R}} e^{-\frac{(x-y)^2}{4(t-s)}} Z(y,s)dyds, \qquad \forall\,(x,t) \in \mathbf{R} \times [0,\infty),$$

satisfies the stochastic heat equation. Show that f has the following covariance for any $s,t > 0$:

$$\mathbf{E}[f(0,s)f(0,t)] = \frac{1}{2\sqrt{\pi}}(|s+t|^{1/2} - |s-t|^{1/2}).$$