Please provide complete and well-written solutions to the following exercises.

Due May 30, in the discussion section.

Homework 7

Exercise 1. Let Z be a standard normal random variable. Recall that we can express a geometric Brownian motion as

$$S(t) = S_0 e^{\sigma\sqrt{t}Z+(r-\sigma^2/2)t}, \qquad t > 0.$$

Show that

$$e^{-rt}\mathbf{E}[S(t)\cdot Z \cdot 1_{\{S(t)>k\}}] = S_0(\Phi'(d_1) + \sigma\sqrt{t}\Phi(d_1)).$$
$$e^{-rt}\mathbf{E}[S(t)\cdot 1_{\{S(t)>k\}}] = S_0\Phi(d_1).$$

Exercise 2. Show the following (using the notation from the Black-Scholes Formula)

- $\Delta = \Phi(d_1)$.
- $\rho = kte^{-rt}\Phi(d_1 - \sigma\sqrt{t})$.
- $\nu = S_0\sqrt{t}\Phi'(d_1)$.
- $-\Theta = \frac{\sigma}{2\sqrt{t}}S_0\Phi'(d_1) + kre^{-rt}\Phi(d_1 - \sigma\sqrt{t})$.

(Hint: use Exercise 1.) (To make these exercises easier, write $c_0 = \mathbf{E}(e^{-rt}\max(S(t)-k,0))$, use the $S(t)$ formula from Exercise 1, and pretend that you can apply the chain rule to the max function, so that $(d/dx)\max(x,0) = 1_{\{x>0\}}$ for any $x \in \mathbf{R}$, even though technically the max function is not differentiable at 0.)

Exercise 3 (MFE Sample Question)**.** You are considering the purchase of a 3-month 41.5-strike American call option on a nondividend-paying stock.

You are given:

(i) The Black-Scholes framework holds.
(ii) The stock is currently selling for 40.
(iii) The stocks volatility is 30%.
(iv) The current call option delta is 0.5.

Determine the current price of the option.

(A) $20 - 20.453\int_{-\infty}^{.15} e^{-x^2/2}dx$.
(B) $20 - 16.138\int_{-\infty}^{.15} e^{-x^2/2}dx$.
(C) $20 - 40.453\int_{-\infty}^{.15} e^{-x^2/2}dx$.
(D) $-20.453 + 16.138\int_{-\infty}^{.15} e^{-x^2/2}dx$.

(E) $-20.453 + 40.453 \int_{-\infty}^{.15} e^{-x^2/2} dx$.

Exercise 4. Let $\mathbf{P}$ be the uniform probability law on $[0,1]$. Let $X(t) = 0$ for any $t \in [0,1]$. For any $n \geq 1$, define $X_n(t) = n \cdot 1_{\{0 \leq t < 1/n\}}$. Show that $X_1, X_2, \ldots$ converges in probability to X. However, $\mathbf{E}X = 0$ whereas $\mathbf{E}X_n = 1$ for all $n \geq 1$. So, convergence in probability does not imply that expected values converge.

Also, note that $X_n(0)$ does not converge to $X(0)$ as $n \to \infty$. So, convergence in probability does not imply pointwise convergence.

Exercise 5 (Uniqueness of the Limit)**.** Suppose $X_1, X_2, \ldots$ converges in probability to X. Also, suppose $X_1, X_2, \ldots$ converges in probability to Y. Show that $\mathbf{P}(X \neq Y) = 0$.

Exercise 6. Let $\{B(t)\}_{t \geq 0}$ be a standard Brownian motion. Let $f \colon \mathbf{R} \to \mathbf{R}$. Assume that $\int_{\mathbf{R}} |f(x)| \, dx < \infty$ and $\int_{\mathbf{R}} f(x) dx = 1$. For any $s > 0$, define

$$X(s) := \frac{1}{\sqrt{s}} \int_0^s f(B(t)) dt.$$

Show that $\lim_{s \to \infty} \mathbf{E}X(s) = \sqrt{2/\pi}$. Then, for an optional challenge, show that $\lim_{s \to \infty} \mathbf{E}(X(s))^2 = 1$. (Hint: for the second part, look up the formula for a multivariate normal random variable.)