Please provide complete and well-written solutions to the following exercises.

Due May 23, in the discussion section.

Homework 6

Exercise 1. There are many ways of buying and selling American put and call options on the same underlying asset, in order to make profits while minimizing risk. These strategies are known as **spreads**. (Every put and call option below will be an American option.) Describe the pros and cons of creating each spread specified below.

- In the **collar** spread, you own a stock which has a variable price s, you buy a put option for that same stock with strick price k_1, and you short a call option with strike price k_2, where $k_1 < k_2$. So, the revenue you will make by exercising all of these options (and selling the stock) is
$$s + \max(k_1 - s, 0) - \max(s - k_2, 0).$$
Plot this function as a function of s. The **zero-cost collar** occurs when k_1 is equal to the current price of the stock.
- In the **straddle** spread, you buy a call and a put option for the same stock and with the same strike price k. So, the revenue you can make by exercising both options simultaneously is
$$\max(k - s, 0) + \max(s - k, 0).$$
Plot this function as a function of s.
- In the **strangle** spread, you buy a call option with strike price k_1, and you buy a put option with strike price k_2, where $k_1 < k_2$. Plot your revenue from exercising both options simultaneously, as a function of s, the price of the underlying asset.
- Let $c > 0$. In the **butterfly** spread, you buy a call option with strike price k, you short two call options with strike price $k + c$, and you buy a call option with strike price $k + 2c$. Plot your revenue from exercising these options simultaneously, as a function of s, the price of the underlying asset.

Exercise 2. There are many ways to try to value an American Put Option. One way is to emulate the formula for a European Put Option which is exercised at time $0 \leq t \leq t_0$:
$$e^{-(\mu+\sigma^2/2)t}\mathbf{E}\max(k - S(t), 0)$$
We would like to simply take the maximum of the above quantity over all $t \in [0, t_0]$. However, this would be equivalent to knowing the future price of the stock at all times, which is unrealistic. So, we instead consider replacing the variable t by a stopping time. Suppose T is a stopping time. That is, $T(t) \geq 0$ is only allowed to depend on values of $S(t')$ where $t' < t$. Then we could try to maximize the quantity
$$e^{-(\mu+\sigma^2/2)T}\mathbf{E}\max(k - S(T), 0)$$

over all stopping times T where $0 \leq T \leq t_0$. To approximate that quantity, let $0 \leq t_1 \leq t_0$ and just consider stopping times T of the form $T = \inf\{t_1 \leq t \leq t_0 \colon S(t) < S(t') \,\forall\, 0 \leq t' \leq (3/4)t_1\}$, or $T = t_0$ if the set of t inside the minimum is empty. Then, using a computer, compute the maximum over all $0 \leq t_1 \leq t_0$ of

$$e^{-(\mu+\sigma^2/2)T}\mathbf{E}\max(k - S(T), 0)$$

for a few selected values of μ, σ, t_0.

This procedure is analogous to the solution of the Secretary Problem.

In order to compute the expected value, use a Monte Carlo simulation of Brownian motion, and take the average value over many runs of the simulation.

Exercise 3. In each of the following examples, choose a few parameters (e.g. use $\mu = 0$, $\sigma = S_0 = t = 1$ and $k = 2$.), and value the option using several runs of a Monte Carlo simulation of Brownian motion. In each case, we multiply by an exponential term in order to emulate the Black-Scholes formula.

(i) (**Asian Call Option**) The value of an Asian option with strike price $k > 0$ at time $t > 0$ is computed using the average value of the stock from time 0 to time t. That is, if the option is exercised at time $t > 0$, then its value is

$$e^{-(\mu+\sigma^2/2)t}\mathbf{E}\max\left((\frac{1}{t}\int_0^t S(r)dr) - k, 0\right).$$

(ii) (**Lookback Call Option**) The value of a lookback call option with strike price $k > 0$ at time $t > 0$ is computed using the maximum value of the stock between time 0 and time t. That is, if the option is exercised at time $t > 0$, then its value is

$$e^{-(\mu+\sigma^2/2)t}\mathbf{E}\max\left(\max_{0\leq r\leq t} S(r) - k, 0\right).$$

In other words, you can "look back" over the past behavior of the stock, and choose the best price possible over the past.

(iii) (**Lookback Put Option**) The value of a lookback put option with strike price $k > 0$ at time $t > 0$ is computed using the minimum value of the stock between time 0 and time t. That is, if the option is exercised at time $t > 0$, then its value is

$$e^{-(\mu+\sigma^2/2)t}\mathbf{E}\max\left(k - \min_{0\leq r\leq t} S(r), 0\right).$$

Finally, using a Corollary from the notes (which gives the CDF of the maximum of Brownian motion), give an exact formula for the value of the Lookback Call Option. (And check that this formula agrees with the results of your simulation.)

Can you also give an explicit formula for the value of the Lookback Put Option?

Exercise 4 (Put-Call Parity for American Options)**.** As we mentioned above, Put-call parity does not hold for American Options, as an equality. However, we can still obtain upper and lower bounds on the difference of the American put and call option, as stated below.

Let c be the price of an American call option for a fixed stock with strike price k, with an option to exercise it at any time $0 \leq t \leq t_0$. Let p be the price of an American put option with strike price k for the same stock, with an option to exercise it at any time $0 \leq t \leq t_0$. Let S_0 be the price of the stock at time 0. Suppose money can be borrowed at a continuously-compounded nominal interest rate $r \geq 0$ (i.e. the rate of interest before adjusting for inflation). Then, assuming no arbitrage opportunity exists,

$$S_0 - k \leq c - p \leq S_0 - ke^{-rt_0}.$$

(Hint: first, show that $p \geq c - S_0 + ke^{-rt_0}$, since p is larger or equal to the value of a European put option, and then apply the Put-Call parity for European options. Then, show that $c \geq p + S_0 - k$ in the following way. Consider the portfolio of buying one call, shorting one put, shorting the stock and borrowing k dollars. If all of the options are exercised at any time $0 \leq t \leq t_0$, show that you obtain a nonnegative profit. That is, the value of this portfolio at time 0 is nonnegative.)

Exercise 5. In the discrete binomial model, we can find a price for an American put option using dynamic programming.

Recall this model. Let $u, d > 0$. Let $0 < p < 1$. Let $(X_1, X_2, \ldots)$ be independent random variables such that $\mathbf{P}(X_n = \log u) =: p$ and $\mathbf{P}(X_n = \log d) = 1 - p \;\forall\; n \geq 1$. Let X_0 be a fixed constant. Let $Y_n := X_0 + \cdots + X_n$, and let $S_n := e^{Y_n} \;\forall\; n \geq 1$. Let $r := p(u-d) - 1 + d$. For any $n \geq 1$, define $M_n := (1+r)^{-n} S_n$. Recall that $M_0, M_1, \ldots$ is a martingale.

Note that, at time n, the random variable S_n has $n+1$ possible values. Label these values as $S_{n,1} \leq \cdots \leq S_{n,m}$. Let $k > 0$. Let $V_{n,m}$ be the value of the American put option at time $n > 0$ with strike price k, when S_n has its m^{th} value. Then

$$V_{n,m} = \max\Big(\max(k - S_{n,m}, 0)\,,\, (1+r)^{-1}(pV_{n+1,m+1} + (1-p)V_{n+1,m})\Big), \qquad \forall\, 1 \leq m \leq n+1.$$

This recursion formula holds since, at step n, you can either exercise the option at time n, or you can wait and see what happens at time $n+1$. The quantity $\max(k - S_{n,m}, 0)$ is your revenue from exercising at time n, and the second quantity $(1+r)^{-1}(pV_{n+1,m+1} + (1-p)V_{n+1,m})$ is your expected revenue from waiting until time $n+1$ to exercise the option. So, at time n, you choose the maximum of these two quantities.

Let's solve this recursion in the following example. Suppose $S_0 = 8$, $p = 1/2$, $u = 2$, $d = 1/2$ (so that $r = 1/4$), and $k = 10$. And suppose the option expires at time $n = 3$ (so that $V_{3,m} = \max(k - S_{3,m}, 0)$ is known for each $1 \leq m \leq 4$.) Then, working backwards, eventually find $V_{0,1}$, the price of the option.

Compare your result in this example with the price of the European put option with the same parameters. (It should be smaller.)