Please provide complete and well-written solutions to the following exercises.

Due May 9, in the discussion section.

Homework 4

Exercise 1 (Scaling Invariance)**.** Let $a > 0$. Let $\{B(t)\}_{t\geq 0}$ be a standard Brownian motion. For any $t > 0$, define $X(t) := \frac{1}{\sqrt{a}}B(at)$. Then $\{X(t)\}_{t\geq 0}$ is also a standard Brownian motion.

Exercise 2. Let $x_1, \ldots, x_n \in \mathbf{R}$, and if $t_n > \cdots > t_1 > 0$. Using the independent increment property, show that the event

$$\{B(t_1) = x_1, \ldots, B(t_n) = x_n\}$$

has a multivariate normal distribution. That is, the joint density of $(B(t_1), \ldots, B(t_n))$ is

$$f(x_1, \ldots, x_n) = f_{t_1}(x_1) f_{t_2 - t_1}(x_2 - x_1) \cdots f_{t_n - t_{n-1}}(x_n - x_{n-1})$$

where

$$f_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/(2t)}, \qquad \forall x \in \mathbf{R},\ t > 0.$$

Exercise 3. Let X be a Gaussian random variable with mean 0 and variance $\sigma_X^2 > 0$. Let Y be a Gaussian random variable with mean 0 and variance $\sigma_Y^2 > 0$. Assume that X and Y are independent. Show that $X + Y$ is also a Gaussian random variable with mean 0 and variance $\sigma_X^2 + \sigma_Y^2$.

(Hint: write an expression for $\mathbf{P}(X + Y \leq t)$, $t \in \mathbf{R}$, then take a derivative in t.)

Exercise 4. Let $\{B(t)\}_{t\geq 0}$ be a standard Brownian motion. Then $\{(B(t))^2 - t\}_{t\geq 0}$ is a (continuous-time) martingale in the following sense: it $t > s > 0$, and if $s > s_n > \cdots > s_1 > 0$, and $x_1, \ldots, x_n \in \mathbf{R}$, then

$$\mathbf{E}((B(t))^2 - t - ((B(s))^2 - s) \,|\, B(s_n) = x_n, \ldots, B(s_1) = x_1) = 0.$$

More generally, for any $\alpha \in \mathbf{R}$, let $Y(t) := e^{\alpha B(t) - \alpha^2 t/2}$. Show that $\{Y(t)\}_{t\geq 0}$ is a martingale.

Then, using the power series expansion of the exponential function, we have $Y(t) = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} M_n(t)$ for some random variables $M_1(t), M_2(t), \ldots$, for any $\alpha \in \mathbf{R}$. It follows that $\{M_1(t)\}_{t\geq 0}$ is a martingale, $\{M_2(t)\}_{t\geq 0}$ is a martingale, etc. (Starting with the following sentence, you do not have to prove anything.) It turns out that

$$M_n(t) = t^{n/2} p_n(B(t)/\sqrt{t}), \quad \forall\, t \in \mathbf{R}, \quad \forall\, n \geq 1,$$

where p_n is the n^{th} Hermite polynomial, so that

$$p_n(x) = e^{x^2/2} (-1)^n \frac{d^n}{dx^n} e^{-x^2/2}, \quad \forall\, x \in \mathbf{R}, \quad \forall\, n \geq 1.$$

For example, using $n = 3$, we know that $\{(B(t))^3 - 3B(t)\}_{t\geq 0}$ is a martingale.

Exercise 5. Let $\{B(t)\}_{t\geq 0}$ be a standard Brownian motion.

- Given that $B(1) = 10$, what is the expected length of time after $t = 1$ until $B(t)$ hits either 8 or 12?
- Now, let $\sigma = 2$, and $\mu = -5$. Suppose a commodity has price $X(t) = \sigma B(t) + \mu t$ for any time $t \geq 0$. Given that the price of the commodity is 4 at time $t = 8$, what is the probability that the price is below 1 at time $t = 9$?
- Suppose a stock has a price $S(t) = 4e^{B(t)}$ for any $t \geq 0$. That is, the stock moves according to Geometric Brownian Motion. What is the probability that the stock reaches a price of 7 before it reaches a price of 2?

Exercise 6. Fix $x > 0$

- Show the bound $\mathbf{P}(-x < B(t) < x) \geq \frac{x}{20\sqrt{t}}$ holds for all $t > x^2$.
- Show that $\mathbf{E}T_x = \infty$.

Exercise 7. Let $\{X(s)\}_{s\geq 0}$ be a standard Brownian motion with drift μ and variance σ^2. For any $t > s > 0$, show that $X(t) - X(s)$ is a Gaussian random variable with mean $\mu(t-s)$ and variance $\sigma^2(t-s)$.

Exercise 8. Let $\{X(t)\}_{t\geq 0} = \{\sigma B(t) + \mu t\}_{t\geq 0}$ be a standard Brownian motion with variance $\sigma^2 > 0$ and drift $\mu \in \mathbf{R}$. Fix $\lambda \in \mathbf{R}$. Then $\{Y(t)\}_{t\geq 0} = \{e^{\lambda X(t) - (\lambda\mu + \lambda^2\sigma^2/2)t}\}_{t\geq 0}$ is a (continuous-time) martingale in the following sense: it $t > s > 0$, and if $s > s_n > \cdots > s_1 > 0$, and $x_1, \ldots, x_n \in \mathbf{R}$, then

$$\mathbf{E}(Y(t) - Y(s) \mid B(s_n) = x_n, \ldots, B(s_1) = x_1) = 0.$$