Please provide complete and well-written solutions to the following exercises.

Due May 2, in the discussion section.

Homework 3

Exercise 1 (Binomial Option Pricing Model)**.** Let $u,d>0$. Let $0<p<1$. Let $(X_1,X_2,\ldots)$ be independent random variables such that $\mathbf{P}(X_n=\log u)=:p$ and $\mathbf{P}(X_n=\log d)=1-p$ $\forall\, n\geq 1$. Let X_0 be a fixed constant. Let $Y_n:=X_0+\cdots+X_n$, and let $S_n:=e^{Y_n}$ $\forall\, n\geq 1$. In general, $S_0,S_1,\ldots$ will not be a martingale, but we can still compute $\mathbf{E}S_n$, by modifying $S_0,S_1,\ldots$ to be a martingale.

First, note that if $n\geq 1$, then Y_n has a binomial distribution, in the sense that

$$\mathbf{P}(Y_n=X_0+i\log u+(n-i)\log d)=\binom{n}{i}p^i(1-p)^{n-i},\qquad \forall\, 0\leq i\leq n.$$

Now define

$$r:=p(u-d)-1+d.$$

Here we chose r so that $p=\frac{1+r-d}{u-d}$. For any $n\geq 1$, define

$$M_n:=(1+r)^{-n}S_n.$$

Show that $M_0,M_1,\ldots$ is a martingale with respect to $X_0,X_1,\ldots$. Consequently,

$$(1+r)^{-n}\mathbf{E}S_n=\mathbf{E}S_0,\qquad \forall\, n\geq 0.$$

(This presentation might be a bit backwards from the financial perspective. Typically, r is a fixed interest rate, and then you choose p such that $p=\frac{1+r-d}{u-d}$. That is, you adjust how the random variables behave in order to get a martingale.)

Exercise 2 (MFE Sample Question)**.** For a two-period binomial model (i.e. the binomial option pricing model with $n=2$), you are given:

(i) Each period is one year.
(ii) The current price for a nondividend-paying stock is 20.
(iii) $u=1.2840$.
(iv) $d=0.8607$.
(v) The continuously compounded risk-free interest rate is 5%. (That is, $1+r=e^{.05}$.)

Calculate the price of an American call option on the stock with a strike price of 22. (That is, compute $(1+r)^{-2}\mathbf{E}\max(S_2-22,0)$. Here S_n is the stock price at time n.)

Exercise 3. Let $X_0=0$. Let $(X_0,X_1,\ldots)$ such that $\mathbf{P}(X_i=1)=\mathbf{P}(X_i=-1)=1/2$ for all $i\geq 1$. For any $n\geq 0$, let $Y_n=X_0+\cdots+X_n$. So, $(Y_0,Y_1,\ldots)$ is a symmetric simple random walk on $\mathbf{Z}$. Show that Y_n^2-n is a martingale (with respect to $(X_0,X_1,\ldots)$).

Exercise 4. Let $1/2 < p < 1$. Let $(X_0, X_1, \ldots)$ such that $\mathbf{P}(X_i = 1) = p$ and $\mathbf{P}(X_i = -1) = 1 - p$ for all $i \geq 1$. For any $n \geq 0$, let $Y_n = X_0 + \cdots + X_n$. Let $T_0 = \min\{n \geq 1 \colon Y_n = 0\}$. If $X_0 = 1$, prove that $\mathbf{P}(T_0 = \infty) > 0$. Then, deduce that, if $X_0 = 0$, then $\mathbf{P}(T_0 = \infty) > 0$. That is, there is a positive probability that the biased random walk never returns to 0, even though it started at 0.

Exercise 5. Let $X_1, \ldots$ be independent identically distributed random variables with $\mathbf{P}(X_i = 1) = \mathbf{P}(X_i = -1) = 1/2$ for every $i \geq 1$. For any $n \geq 1$, let $M_n := X_1 + \cdots + X_n$. Let $M_0 = 0$. For any $n \geq 1$, define

$$W_n := M_0 + \sum_{m=1}^{n} H_m (M_m - M_{m-1}).$$

Show that if you have an infinite amount of money, then you *can* make money by using the double-your-bet strategy in the game of coinflips (where if you bet \$$d$, then you win \$$d$ with probability 1/2, and you lose \$$d$ with probability 1/2). For example, show that if you start by betting \$1, and if you keep doubling your bet until you win (which should define some betting strategy $H_1, H_2, \ldots$ and a stopping time T), then $\mathbf{E}W_T = 1$, for a suitable stopping time T.

Exercise 6. Prove the following variant of the Optional Stopping Theorem. Assume that $(M_0, M_1, \ldots)$ is a submartingale, and let T be a stopping time such that $\mathbf{P}(T < \infty) = 1$. Let $c \in \mathbf{R}$. Assume that $|M_{n \wedge T}| \leq c$ for all $n \geq 0$. Then $\mathbf{E}M_T \geq \mathbf{E}M_0$. That is, you can make money by stopping a submartingale.

Exercise 7 (Ballot Theorem)**.** Let a, b be positive integers. Suppose there are c votes cast by c people in an election. Candidate 1 gets a votes and candidate 2 gets b votes. (So $c = a + b$.) Assume $a > b$. The votes are counted one by one. The votes are counted in a uniformly random ordering, and we would like to keep a running tally of who is currently winning. (News agencies seem to enjoy reporting about this number.) Suppose the first candidate eventually wins the election. We ask: with what probability will candidate 1 always be ahead in the running tally of who is currently winning the election? As we will see, the answer is $\frac{a-b}{a+b}$.

To prove this, for any positive integer k, let S_k be the number of votes for candidate 1, minus the number of votes for candidate 2, after k votes have been counted. Then, define $X_k := S_{c-k}/(c-k)$. Show that $X_0, X_1, \ldots$ is a martingale (with respect to $S_c, S_{c-1}, \ldots$). Then, let T such that $T = \min\{0 \leq k \leq c \colon X_k = 0\}$, or $T = c - 1$ if no such k exists. Apply the Optional Stopping theorem to X_T to deduce the result.

Exercise 8. Let $(X_0, X_1, \ldots)$ be the simple random walk on $\mathbf{Z}$. For any $n \geq 0$, define $M_n = X_n^3 - 3nX_n$. Show that $(M_0, M_1, \ldots)$ is a martingale with respect to $(X_0, X_1, \ldots)$

Now, fix $m > 0$ and let T be the first time that the walk hits either 0 or m. Show that, for any $0 < k \leq m$, if $X_0 = k$, then

$$\mathbf{E}(T \mid X_T = m) = \frac{m^2 - k^2}{3}.$$

(You can apply the Optional stopping theorem without verifying that the martingale is bounded.)

Exercise 9. Let $X_1, X_2, \ldots$ be independent random variables with $\mathbf{E}X_i = 0$ for every $i \geq 1$. Suppose there exists $\sigma > 0$ such that $\mathrm{Var}(X_i) = \sigma^2$ for all $i \geq 1$. For any $n \geq 1$, let $S_n = X_1 + \cdots + X_n$. Show that $S_n^2 - n\sigma^2$ is a martingale with respect to $X_1, X_2, \ldots$. (We let $X_0 = 0$.)

Let $a > 0$. Let $T = \min\{n \geq 1 \colon |S_n| \geq a\}$. Using the Optional Stopping Theorem, show that $\mathbf{E}T \geq a^2/\sigma^2$. Observe that a simple random walk on $\mathbf{Z}$ has $\sigma^2 = 1$ and $\mathbf{E}T = a^2$ when $a \in \mathbf{Z}$.

(When applying the Optional Stopping Theorem, you do not have to show that the martingale is bounded.)