Please provide complete and well-written solutions to the following exercises.

Due April 18, in the discussion section.

Homework 2

Exercise 1 (Confidence Intervals)**.** Among 625 members of a bank chosen uniformly at random among all bank members, it was found that 25 had a savings account. Give an interval of the form $[a, b]$ where $0 \leq a, b \leq 625$ are integers, such that with about 95% certainty, the number of any set of 625 bank members with savings accounts chosen uniformly at random lies in the interval $[a, b]$. (Hint: if Y is a standard Gaussian random variable, then $\mathbf{P}(-2 \leq Y \leq 2) \approx .95$.)

Exercise 2 (Hypothesis Testing)**.** Suppose we run a casino, and we want to test whether or not a particular roulette wheel is biased. Let p be the probability that red results from one spin of the roulette wheel. Using statistical terminology, "$p = 18/38$" is the null hypothesis, and "$p \neq 18/38$" is the alternative hypothesis. (On a standard roulette wheel, 18 of the 38 spaces are red.) For any $i \geq 1$, let $X_i = 1$ if the i^{th} spin is red, and let $X_i = 0$ otherwise.

Let $\mu := \mathbf{E}X_1$ and let $\sigma := \sqrt{\mathrm{var}(X_1)}$. If the null hypothesis is true, and if Y is a standard Gaussian random variable

$$\lim_{n\to\infty} \mathbf{P}\left(\left|\frac{X_1 + \cdots + X_n - n\mu}{\sigma\sqrt{n}}\right| \geq 2\right) = \mathbf{P}(|Y| \geq 2) \approx .05.$$

To test the null hypothesis, we spin the wheel n times. In our test, we reject the null hypothesis if $|X_1 + \cdots + X_n - n\mu| > 2\sigma\sqrt{n}$. Rejecting the null hypothesis when it is true is called a type I error. In this test, we set the type I error percentage to be 5%. (The type I error percentage is closely related to the p-value.)

Suppose we spin the wheel $n = 3800$ times and we get red 1868 times. Is the wheel biased? That is, can we reject the null hypothesis with around 95% certainty?

Exercise 3 (Numerical Integration)**.** In computer graphics in video games, etc., various integrations are performed in order to simulate lighting effects. Here is a way to use random sampling to integrate a function in order to quickly and accurately render lighting effects. Let $\Omega = [0, 1]$, and let $\mathbf{P}$ be the uniform probably law on Ω, so that if $0 \leq a < b \leq 1$, we have $\mathbf{P}([a, b]) = b - a$. Let $X_1, \ldots, X_n$ be independent random variables such that $\mathbf{P}(X_i \in [a, b]) = b - a$ for all $0 \leq a < b \leq 1$, for all $i \in \{1, \ldots, n\}$. Let $f : [0, 1] \to \mathbf{R}$ be a continuous function we would like to integrate. Instead of integrating f directly, we instead compute the quantity

$$\frac{1}{n}\sum_{i=1}^{n} f(X_i).$$

Show that

$$\lim_{n\to\infty} \mathbf{E}\left(\frac{1}{n}\sum_{i=1}^{n} f(X_i)\right) = \int_0^1 f(t)dt.$$

$$\lim_{n\to\infty} \mathrm{var}\left(\frac{1}{n}\sum_{i=1}^{n} f(X_i)\right) = 0.$$

That is, as n becomes large, $\frac{1}{n}\sum_{i=1}^{n} f(X_i)$ is a good estimate for $\int_0^1 f(t)dt$.

Exercise 4 (Optional; Numerical Integration, Continued)**.** Let $\mathbf{P}$ denote the uniform probability law on $[0,1]$, and let $X\colon [0,1]\to\mathbf{R}$ be a random variable. This exercise discusses how to numerically compute expected values on a computer, as in Exercise 3. The procedure below is an example of Monte Carlo simulation.

Consider the function $X(t) := t$ for all $t\in[0,1]$. We know that $\mathbf{E}X = 1/2$. To approximate $\mathbf{E}X$ with Matlab, we can use `sum(rand(1,1000))/1000`, which sums 1000 independent, random samples from the uniform probability law on $[0,1]$, and averages them (by dividing by 1000). Enter the term `sum(rand(1,1000))/1000` a few times in the command line of Matlab, to get a few different results.

Consider the function $X(t) := t^2$ for all $t\in[0,1]$. Using Matlab, approximate $\mathbf{E}X$ by averaging 1000 random samples from the uniform probability law on $[0,1]$.

Now, let $\mathbf{P}$ denote the standard Gaussian probability law on $\mathbf{R}$, so that

$$\mathbf{E}X := \int_{-\infty}^{\infty} X(t)e^{-t^2/2}dt/\sqrt{2\pi}$$

for any function $X\colon\mathbf{R}\to\mathbf{R}$. Using the Matlab function `randn`, approximate $\mathbf{E}X$ for $X(t) := t$ and $X(t) := t^2$ by averaging 1000 random samples from the standard Gaussian probability law.

Remark 1. When Matlab or other computer programs generate "random numbers" using e.g. `rand` or `randn`, these numbers are not actually random or independent. These numbers are pseudorandom. That is, functions such as `rand` output numbers in a deterministic way, but these numbers behave as if they were random. All "random" numbers generated by computers are actually pseudorandom, and this includes slot machines at casinos, video games, etc. So, when using Monte Carlo simulation as we did above, we should be careful about interpreting our results, since it is generally impossible to take random samples from a probability law on a computer.

And, theoretically, if you knew enough about the random number generator that a slot machine is using, you could predict its output.

Remark 2. You may believe that you could create a random sequence of numbers by pressing buttons on your keyboard, therein bypassing any need for a pseudorandom number generator. The website at the link here may convince you otherwise.

Exercise 5. Suppose you begin at the lower left corner of an 8×8 chess board. Every day, you are allowed to move either up or right to a consecutive board space (unless you are waiting). When you land on a new space, you have to wait a number of days specified by the number sitting on that board space, until you move again. The numbers on the board spaces appear below.

$$\begin{pmatrix} 1 & 2 & 2 & 1 & 3 & 2 & 6 & 0 \\ 4 & 7 & 3 & 2 & 4 & 8 & 3 & 4 \\ 3 & 4 & 4 & 4 & 5 & 5 & 4 & 2 \\ 4 & 7 & 5 & 3 & 4 & 4 & 5 & 5 \\ 4 & 5 & 4 & 2 & 3 & 3 & 7 & 3 \\ 4 & 6 & 6 & 4 & 3 & 4 & 3 & 2 \\ 5 & 4 & 6 & 3 & 4 & 3 & 4 & 1 \\ 0 & 3 & 6 & 2 & 7 & 2 & 7 & 5 \end{pmatrix}.$$

Your goal is to reach the top right corner of the chess board in the shortest amount of time. Find the path that takes the shortest amount of time, and also find the shortest amount of time that it takes to reach the top right corner. (Hint: Use recursion. That is, solve a more general problem. For *any* square on the board, find the least number of days it takes to reach that square starting from the bottom left corner, using only up and right moves. If you are still stuck, read a bit about dynamic programming.)

Exercise 6. Let $\Omega = [0,1]$. Let $\mathbf{P}$ be the uniform probability law on Ω. Let $X: [0,1] \to \mathbf{R}$ be a random variable such that $X(t) = t^2$ for all $t \in [0,1]$. Let

$$\mathcal{A} = \{[0,1/4), [1/4,1/2), [1/2,3/4), [3/4,1]\}$$

Compute explicitly the function $\mathbf{E}(X|\mathcal{A})$. (It should be constant on each of the partition elements.) Draw the function $\mathbf{E}(X|\mathcal{A})$ and compare it to a drawing of X itself.

Now, for every integer $k > 1$, let $s = 2^{-k}$, and let $\mathcal{A}_k := \{[0,s), [s,2s), [2s,3s), \ldots, [1-2s, 1-s), [1-s,1)\}$. Try to draw $\mathbf{E}(X|\mathcal{A}_k)$. Convince yourself of the following fact (you can prove it if you want, but you do not have to): for every $t \in [0,1]$

$$\lim_{k\to\infty} \mathbf{E}(X|\mathcal{A}_k)(t) = X(t).$$

The purpose of this exercise is to demonstrate that $\mathbf{E}(X|\mathcal{A})$ is given by averaging X over each partition element, such that $\mathbf{E}(X|\mathcal{A})$ is constant on each partition element of $\mathcal{A}$.

Exercise 7. Let X be a random variable with finite variance, and let $t \in \mathbf{R}$. Consider the function $f: \mathbf{R} \to \mathbf{R}$ defined by $f(t) = \mathbf{E}(X-t)^2$. Show that the function f is uniquely minimized when $t = \mathbf{E}X$. That is, $f(\mathbf{E}X) < f(t)$ for all $t \in \mathbf{R}$ such that $t \neq \mathbf{E}X$. Put another way, setting t to be the mean of X minimizes the quantity $\mathbf{E}(X-t)^2$ uniquely.

The conditional expectation, being a piecewise version of taking an average, has a similar property. Let $A_1, \ldots, A_k \subseteq \Omega$ such that $A_i \cap A_j = \emptyset$ for all $i, j \in \{1, \ldots, k\}$ with $i \neq j$, and $\cup_{i=1}^k A_i = \Omega$. Write $\mathcal{A} = \{A_1, \ldots, A_k\}$. By definition, for each $1 \leq i \leq k$, $\mathbf{E}(X|\mathcal{A})$ is constant on A_i. Now, let Y be any other random variable such that, for each $1 \leq i \leq k$, Y is constant on A_i. Show that the quantity $\mathbf{E}(X-Y)^2$ is uniquely minimized by such a Y only when $Y = \mathbf{E}(X|\mathcal{A})$.

Exercise 8. Let $\Omega = [0,1]$. Let $\mathbf{P}$ be the uniform probability law on Ω. Let $X: [0,1] \to \mathbf{R}$ be a random variable such that $X(t) = t^2$ for all $t \in [0,1]$. For every integer $k > 1$, let $s = 2^{-k}$, let $\mathcal{A}_k := \{[0,s), [s,2s), [2s,3s), \ldots, [1-2s, 1-s), [1-s,1)\}$, and let $M_k := \mathbf{E}(X|\mathcal{A}_k)$. Show that the increments $M_2 - M_1, M_3 - M_2, \ldots$ are orthogonal in the following sense. For any $i, j \geq 1$ with $i \neq j$,

$$\mathbf{E}(M_{i+1} - M_i)(M_{j+1} - M_j) = 0.$$

This property is sometimes called **orthogonality of martingale increments**. This property holds for many martingales, but we will not prove this.