Math 174E, Spring 2017, UCLA — Instructor: Steven Heilman

Name: ______________________ UCLA ID: ______________ Date: ___________

Signature: ______________________.
(By signing here, I certify that I have taken this test while refraining from cheating.)

Mid-Term 2

This exam contains 8 pages (including this cover page) and 5 problems. Check to see if any pages are missing. Enter all requested information on the top of this page.

You may *not* use your books, notes, or any calculator on this exam. You are required to show your work on each problem on this exam. The following rules apply:

- You have 50 minutes to complete the exam, starting at the beginning of class.
- **If you use a theorem or proposition from class or the notes or the book you must indicate this** and explain why the theorem may be applied. It is okay to just say, "by some theorem/proposition from class."
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this. Scratch paper appears at the end of the document.

Do not write in the table to the right. Good luck![a]

Problem	Points	Score
1	12	
2	12	
3	6	
4	10	
5	10	
Total:	50	

Reference sheet

Below are some definitions that may be relevant.

Standard Brownian motion is a stochastic process $\{B(t)\}_{t\geq 0}$ with $B(0) = 0$ such that:

(i) (Continuous Sample Paths) With probability 1, the function $t \mapsto B(t)$ is continuous.

(ii) (Stationary Gaussian increments) for any $0 < s < t$, $B(t) - B(s)$ is a Gaussian random variable with mean zero and variance $t - s$.

(iii) (Independent increments) For any $0 < t_1 < \cdots < t_n$, the random variables $B(t_2) - B(t_1), \ldots, B(t_n) - B(t_{n-1})$ are all independent.

Let $S_0 > 0$, $\sigma > 0$, $\mu \in \mathbf{R}$. Then geometric Brownian motion is a stochastic process of the form $\{S(t)\}_{t\geq 0} = \{S_0 e^{\sigma B(t) + \mu t}\}_{t\geq 0}$.

The European Call Option with strike price $k > 0$ and expiration time $t > 0$ has payoff $\max(S(t) - k, 0)$. The European Put Option with strike price $k > 0$ and expiration time $t > 0$ has payoff $\max(k - S(t), 0)$.

The value of the European call option with expiration time t and strike price k is

$$c = S_0 \Phi(d_1) - e^{-rt} k \Phi(d_1 - \sigma\sqrt{t}),$$

where

$$d_1 := \frac{\log(S_0/k) + (r + \sigma^2/2)t}{\sigma\sqrt{t}}, \qquad \Phi(d_1) := \int_{-\infty}^{d_1} e^{-y^2/2} \frac{dy}{\sqrt{2\pi}}$$

Let $\{X(t)\}_{t\geq 0}$ be a real-valued continuous-time stochastic process. A **continuous-time martingale with respect to** $\{B(t)\}_{t\geq 0}$ is a stochastic process $\{X(t)\}_{t\geq 0}$ such that for any $m_0, x_0, \ldots, x_n \in \mathbf{R}$, and for any $t > s > s_n > \cdots > s_1 > 0$,

$$\mathbf{E}(X(t) - X(s) \,|\, B(s_n) = x_n, \ldots, B(s_1) = x_1, X(0) = m_0) = 0.$$

1. (12 points) Let $\{B(t)\}_{t\geq 0}$ be a standard Brownian motion.

- Let $t > s > 0$. Show that $\mathbf{E}B(s)B(t) = s$.
- Let $a, b > 0$. Let $T_a := \min\{t \geq 0\colon B(t) = a\}$. Show that

$$\mathbf{P}(T_a < T_{-b}) = \frac{b}{a+b}$$

(If you use the Optional Stopping Theorem, then the only assumption you need to verify is that you have a martingale.)

2. (12 points) Let $\{B(t)\}_{t\geq 0}$ be a standard Brownian motion.

- Show that $\{(B(t))^2 - t\}_{t\geq 0}$ is a (continuous-time) martingale.
- Let $a, b > 0$. Let $T = \min\{t \geq 0 \colon B(t) \notin (-b, a)\}$. Show that

$$\mathbf{E}T = ab.$$

(If you use the Optional Stopping Theorem, then the only assumption you need to verify is that you have a martingale.)

3. (6 points) Consider a European call option and a European put option on a nondividend-paying stock. The following things are given

   - The current price of the stock is 60.
   - The call option currently sells for 0.15 more than the put option.
   - Both the call option and put option will expire in 4 years.
   - Both the call option and put option have a strike price of 70.

   Calculate the continuously compounded risk-free interest rate. (That is, compute the interest rate r that ensures that no arbitrage opportunity exists.)

4. (10 points) The value of a lookback call option with strike price $k > 0$ at time $t > 0$ is

$$e^{-(\mu+\sigma^2/2)t}\mathbf{E}\max\left(\max_{0\le r\le t} S(r) - k, 0\right).$$

Here $\{S(t)\}_{t\ge 0}$ is a geometric Brownian motion.

Write down an explicit integral formula for $\mathbf{E}\max\left(\max_{0\le r\le t} S(r) - k, 0\right)$.

5. (10 points) In this problem, we consider the price of the **knockout option** using the binomial model.

Recall the binomial model. Let $u, d > 0$. Let $0 < p < 1$. Let $(X_1, X_2, \ldots)$ be independent random variables such that $\mathbf{P}(X_n = \log u) =: p$ and $\mathbf{P}(X_n = \log d) = 1 - p \ \forall\ n \geq 1$. Let X_0 be a fixed constant. Let $Y_n := X_0 + \cdots + X_n$, and let $S_n := e^{Y_n} \ \forall\ n \geq 1$. Let $r := p(u - d) - 1 + d$. For any $n \geq 1$, define $M_n := (1 + r)^{-n} S_n$. Recall that $M_0, M_1, \ldots$ is a martingale, and S_n is the price of the stock at time n.

You are required to compute the price of a call option with strike price 28 and a knockout barrier of 20. We assume that $S_0 = 24$, $u = 3/2$, $d = 2/3$, $r = 1/6$, so that $p = 3/5$. The option expires at time $n = 3$, and you have the ability to exercise the option only at time $n = 3$. At any time $1 \leq n \leq 3$, if the price of the stock has dropped below the barrier value of 20, then the option becomes worthless at the current time and at any future time. Otherwise, if the stock has a current price s, then the option has a payoff of $\max(s - 28, 0)$.

(Scratch paper)