174E Midterm 1 Solutions, Spring 2017[1]

1. Question 1

Label the following statements as TRUE or FALSE. If the statement is true, **explain your reasoning**. If the statement is false, **provide a counterexample and explain your reasoning**.

(a) For any positive integers i,j, let a_{ij} be a real number. Then

$$\sum_{i=1}^{\infty}\left(\sum_{j=1}^{\infty}a_{ij}\right)=\sum_{j=1}^{\infty}\left(\sum_{i=1}^{\infty}a_{ij}\right).$$

FALSE. For any $i\geq 1$, let $a_{i(i+1)}=1$, let $a_{ii}=-1$, and let $a_{ij}=0$ for any other i,j. Then $\sum_{i=1}^{\infty}\left(a_{ii}+a_{i(i+1)}\right)=\sum_{i=1}^{\infty}(0)=0\neq -1=a_{11}+0=a_{11}+\sum_{j=2}^{\infty}\left(a_{(j-1)j}+a_{jj}\right)=\sum_{j=1}^{\infty}\left(\sum_{i=1}^{\infty}a_{ij}\right)$

(b) Let $M_0=0$ and let $M_0,M_1,\ldots$ be a martingale. Let T be a stopping time for the martingale. Then $\mathbb{E}M_T=0$.

FALSE. Consider the simple random walk on the integers, and let $T:=\min\{n\geq 1\colon M_n=1\}$. Then $M_T=1$ so $\mathbb{E}M_T=1\neq 0$.

(c) Let X,Y be discrete random variables such that

$$\mathbf{P}(X\leq x,Y=y)=\mathbf{P}(X\leq x)\mathbf{P}(Y=y),\qquad \forall\,x,y\in\mathbb{R}.$$

Then

$$\mathbf{P}(X\leq x,Y\leq y)=\mathbf{P}(X\leq x)\mathbf{P}(Y\leq y),\qquad \forall\,x,y\in\mathbb{R}.$$

TRUE. For any $t\in\mathbb{R}$, let $A_t=\{Y=t\}$. Then $A_{t_1}\cap A_{t_2}=\emptyset$ if $t_1\neq t_2$, and $\cup_{t\leq y}A_t=\{Y\leq y\}$, so

$$\sum_{t\leq y}\mathbf{P}(X\leq x,Y=t)=\sum_{t\leq y}\mathbf{P}(\{X\leq x\}\cap A_t)=\mathbf{P}(\{X\leq x\}\cap(\cup_{t\leq y}A_t))=\mathbf{P}(X\leq x,Y\leq y).$$

Similarly, $\sum_{t\leq y}\mathbf{P}(Y=t)=\mathbf{P}(Y\leq y)$. So, summing both sides of the equality $\mathbf{P}(X\leq x,Y=t)=\mathbf{P}(X\leq x)\mathbf{P}(Y=t)$ over all $t\leq y$ proves the assertion.

2. Question 2

For any $x\in\mathbb{R}$, define

$$\phi(x):=\max\Big(-x-1,\,0,\,x-1\Big).$$

Prove that $\phi\colon\mathbb{R}\to\mathbb{R}$ is convex.

(In this problem, unlike the other problems, you are allowed to use results from the homework.)

Solution. From Homework 1, Exercise 1, it suffices to show: for any $y\in\mathbb{R}$, there exists $a\in\mathbb{R}$ such that $L(x):=a(x-y)+\phi(y)$ satisfies $L(y)=\phi(y)$ and $L(x)\leq\phi(x)$ for all $x\in\mathbb{R}$. We break into three cases.

Case 1. $y\in[-1,1]$. In this case we choose $a=0$. Then $L(x)=0$ for all $x\in\mathbb{R}$, $L(y)=\phi(y)=0$, and $L(x)=0\leq\phi(x)$ for all $x\in\mathbb{R}$ by definition of ϕ.

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Case 2. $y > 1$. In this case we choose $a = 1$. Then $L(x) = (x - y) + (y - 1) = x - 1$ for all $x \in \mathbb{R}$, $L(y) = y - 1 = \phi(y)$, and $L(x) \leq \phi(x)$ for all $x \in \mathbb{R}$, since $L(x) = \phi(x)$ when $x \geq 1$, and if $x < 1$, then $L(x) < 0 \leq \phi(x)$ since $\phi(x) \geq 0$ by definition of ϕ.

Case 2. $y < -1$. In this case we choose $a = -1$. Then $L(x) = -(x-y)+(-y-1) = -x-1$ for all $x \in \mathbb{R}$, $L(y) = -y - 1 = \phi(y)$, and $L(x) \leq \phi(x)$ for all $x \in \mathbb{R}$, since $L(x) = \phi(x)$ when $x \leq -1$, and if $x > -1$, then $L(x) < 0 \leq \phi(x)$ since $\phi \geq 0$ by definition of ϕ.

3. Question 3

Suppose you begin at the lower left corner of a 4×4 chess board. Every day, you are allowed to move either up or right to a consecutive board space (unless you are waiting). When you land on a new space, you have to wait a number of days specified by the number sitting on that board space, until you move again. The numbers on the board spaces appear below.

$$\begin{pmatrix} 3 & 3 & 7 & 0 \\ 3 & 4 & 3 & 2 \\ 4 & 3 & 4 & 1 \\ 0 & 2 & 7 & 5 \end{pmatrix}.$$

Your goal is to reach the top right corner of the chess board in the shortest amount of time. Find the path that takes the shortest amount of time, and also find the shortest amount of time that it takes to reach the top right corner.

Solution. We begin in the bottom left corner, and inductively label each space by the shortest amount of time it takes to reach that space, using only up and right moves.

We show a few stages of this procedure.

$$\begin{pmatrix} 3 & 3 & 7 & 0 \\ 7 & 4 & 3 & 2 \\ 4 & 5 & 4 & 1 \\ 0 & 2 & 9 & 5 \end{pmatrix}, \quad \begin{pmatrix} 10 & 3 & 7 & 0 \\ 7 & 9 & 3 & 2 \\ 4 & 5 & 9 & 1 \\ 0 & 2 & 9 & 14 \end{pmatrix}, \quad \begin{pmatrix} 10 & 12 & 7 & 0 \\ 7 & 9 & 12 & 2 \\ 4 & 5 & 9 & 10 \\ 0 & 2 & 9 & 14 \end{pmatrix}, \quad \begin{pmatrix} 10 & 12 & 19 & 0 \\ 7 & 9 & 12 & 12 \\ 4 & 5 & 9 & 10 \\ 0 & 2 & 9 & 14 \end{pmatrix}.$$

For any particular space s, the shortest path to s must pass through its left or bottom neighbor. So, for any space s, the shortest path through s is obtained by taking the shortest path through either the left or bottom neighbor of s (whichever is shorter).

So, the shortest number of days needed to reach the top right corner is 12, and this path can be found by retracing the inductive procedure, and following the path of red numbers

$$\begin{pmatrix} 3 & 3 & 7 & 0 \\ 3 & 4 & 3 & 2 \\ 4 & 3 & 4 & 1 \\ 0 & 2 & 7 & 5 \end{pmatrix}, \quad \begin{pmatrix} & & & 0 \\ & & & * \\ & * & * & * \\ 0 & * & & \end{pmatrix}.$$

And indeed, $2 + 3 + 4 + 1 + 2 = 12$.

4. Question 4

Let $X \geq 0$ be a random variable such that $\mathbf{P}(X > 0) > 0$. Show that

$$\mathbb{E}(X \,|\, X > 0) \leq \frac{\mathbb{E}X^2}{\mathbb{E}X}.$$

(Hint: you can freely use the Cauchy-Schwarz inequality: $(\mathbb{E}XY)^2 \leq \mathbb{E}X^2\mathbb{E}Y^2$.)

Solution. Since $X \geq 0$ and $\mathbf{P}(X > 0) > 0$, we know that $\mathbb{E}X > 0$. So, we are required to show that $\mathbb{E}X\mathbb{E}(X|X > 0) \leq \mathbb{E}X^2$. Since $X = X \cdot 1_{X>0}$, we are required to show that $[\mathbb{E}(X \cdot 1_{X>0})]^2/\mathbf{P}(X > 0) \leq \mathbb{E}X^2$. Rearranging, we need to show that $[\mathbb{E}(X \cdot 1_{X>0})]^2 \leq \mathbb{E}X^2\mathbf{P}(X > 0)$. Since $\mathbb{E}1_{X>0}^2 = \mathbb{E}1_{X>0} = \mathbf{P}(X > 0)$, our desired inequality follows from the Cauchy-Schwarz inequality.

5. Question 5

Let $X_0 = 0$, and let $a < 0 < b$ be integers. Let $0 < p < 1$ with $p \neq 1/2$. Let $X_1, X_2, \ldots$ be independent identically distributed random variables so that $\mathbf{P}(X_i = 1) = p$ and $\mathbf{P}(X_i = -1) = 1 - p$ for all $i \geq 1$. For any $n \geq 0$, let $Y_n := X_0 + \cdots + X_n$. Define $T := \min\{n \geq 1 \colon Y_n \notin (a, b)\}$.

Compute $\mathbb{E}T$, in terms of a, b, p.

(Hint: use martingales, somehow. If you use the Optional Stopping Theorem, you do not have to verify that the martingale is bounded.)

(Second hint: you can freely use the formula $\mathbf{P}(Y_T = a) = \frac{(q/p)^{x_0} - (q/p)^b}{(q/p)^a - (q/p)^b}$, where $q := 1 - p$.)

Solution. Let $Z_0 := 0$. For any $n \geq 1$, let $Z_n := Y_n - n\mu$, where $\mu := \mathbb{E}X_1 = p - (1 - p) = 1 - 2p$. As shown in the notes in Example 3.15, $Z_0, Z_1, Z_2, \ldots$ is a martingale. So, from the Optional Stopping Theorem, Version 2,

$$0 = \mathbb{E}Z_0 = \mathbb{E}Z_T = \mathbb{E}Y_T - \mu\mathbb{E}T.$$

That is,

$$\mathbb{E}T = \frac{1}{\mu}\mathbb{E}Y_T = \frac{1}{1 - 2p}(ca + (1 - c)b).$$