Please provide complete and well-written solutions to the following exercises.

Due March 14, in the discussion section.

Homework 8

Exercise 1. Suppose you run a (busy) car wash, and the number of cars that come to the car wash between time 0 and time $s > 0$ is a Poisson poisson with rate $\lambda = 1$. Suppose each car is equally likely to have one, two, three, or four people in it. What is the average number of cars with four people that have arrived by time $s = 100$?

Exercise 2. Let X be a Poisson random variable with parameter $\lambda > 0$. Let Y be a Poisson random variable with parameter $\delta > 0$. Assume that X, Y are independent. Then $X + Y$ is a Poisson random variable with parameter $\lambda + \delta$.

Exercise 3. Suppose you are still running a (busy) car wash. The number of red cars that come to the car wash between time 0 and time $s > 0$ is a Poisson poisson with rate 2. The number of blue cars that come to car wash between time 0 and time $s > 0$ is a Poisson poisson with rate 3. Both Poisson processes are independent of each other. All cars are either red or blue. With what probability will five blue cars arrive, before three red cars have arrived?

Exercise 4 (Scaling Invariance)**.** Let $a > 0$. Let $\{B(t)\}_{t\geq 0}$ be a standard Brownian motion. For any $t > 0$, define $X(t) := \frac{1}{\sqrt{a}}B(at)$. Then $\{X(t)\}_{t\geq 0}$ is also a standard Brownian motion.

Exercise 5. Let $x_1, \ldots, x_n \in \mathbf{R}$, and if $t_n > \cdots > t_1 > 0$. Using the independent increment property, show that the event

$$\{B(t_1) = x_1, \ldots, B(t_n) = x_n\}$$

has a multivariate normal distribution. That is, the joint density of $(B(t_1), \ldots, B(t_n))$ is

$$f(x_1, \ldots, x_n) = f_{t_1}(x_1) f_{t_2 - t_1}(x_2 - x_1) \cdots f_{t_n - t_{n-1}}(x_n - x_{n-1})$$

where

$$f_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/(2t)}, \qquad \forall x \in \mathbf{R},\, t > 0.$$

Exercise 6. Let X be a Gaussian random variable with mean 0 and variance $\sigma_X^2 > 0$. Let Y be a Gaussian random variable with mean 0 and variance $\sigma_Y^2 > 0$. Assume that X and Y are independent. Show that $X + Y$ is also a Gaussian random variable with mean 0 and variance $\sigma_X^2 + \sigma_Y^2$.

(Hint: write an expression for $\mathbf{P}(X + Y \leq t)$, $t \in \mathbf{R}$, then take a derivative in t.)

Exercise 7. Let $A := \{1, 1/2, 1/3, 1/4, 1/5, \ldots\}$. Find $\inf(A)$. Note that $\inf(A)$ exists, but A has no minimum element. The infimum is better to work with for this reason.

Exercise 8. Let $\{B(t)\}_{t\geq 0}$ be a standard Brownian motion. Then $\{(B(t))^2 - t\}_{t\geq 0}$ is a (continuous-time) martingale in the following sense: it $t > s > 0$, and if $s > s_n > \cdots > s_1 > 0$, and $x_1, \ldots, x_n \in \mathbf{R}$, then

$$\mathbf{E}((B(t))^2 - t - ((B(s))^2 - s) \,|\, B(s_n) = x_n, \ldots, B(s_1) = x_1) = 0.$$

Exercise 9. Let $\{B(t)\}_{t\geq 0}$ be a standard Brownian motion.

- Given that $B(1) = 10$, what is the expected length of time after $t = 1$ until $B(t)$ hits either 8 or 12?
- Now, let $\sigma = 2$, and $\mu = -5$. Suppose a commodity has price $X(t) = \sigma B(t) + \mu t$ for any time $t \geq 0$. Given that the price of the commodity is 4 at time $t = 8$, what is the probability that the price is below 1 at time $t = 9$?
- Suppose a stock has a price $S(t) = 4e^{B(t)}$ for any $t \geq 0$. That is, the stock moves according to Geometric Brownian Motion. What is the probability that the stock reaches a price of 7 before it reaches a price of 2?

Exercise 10. Fix $x > 0$

- Show the bound $\mathbf{P}(-x < B(t) < x) \geq \frac{x}{20\sqrt{t}}$ holds for all $t > x^2$.
- Show that $\mathbf{E}T_x = \infty$. (Recall we observed something similar for the simple random walk on $\mathbf{Z}$.)

Exercise 11 (Optional)**.** Let $\{X(s)\}_{s\geq 0}$ be a standard Brownian motion with drift μ and variance σ^2. For any $t > s > 0$, show that $X(t) - X(s)$ is a Gaussian random variable with mean $\mu(t - s)$ and variance $\sigma^2(t - s)$.

Exercise 12 (Optional)**.** Let $\{X(t)\}_{t\geq 0} = \{\sigma B(t) + \mu t\}_{t\geq 0}$ be a standard Brownian motion with variance $\sigma^2 > 0$ and drift $\mu \in \mathbf{R}$. Fix $\lambda \in \mathbf{R}$. Then $\{Y(t)\}_{t\geq 0} = \{e^{\lambda X(t) - (\lambda\mu + \lambda^2\sigma^2/2)t}\}_{t\geq 0}$ is a (continuous-time) martingale in the following sense: it $t > s > 0$, and if $s > s_n > \cdots > s_1 > 0$, and $x_1, \ldots, x_n \in \mathbf{R}$, then

$$\mathbf{E}(Y(t) - Y(s) \,|\, B(s_n) = x_n, \ldots, B(s_1) = x_1) = 0.$$

Exercise 13 (Optional)**.** Let $\{X(t)\}_{t\geq 0} = \{\sigma B(t) + \mu t\}_{t\geq 0}$ be a standard Brownian motion with variance $\sigma^2 > 0$ and negative drift $\mu < 0$. Let $a < 0 < b$. Let $T := \inf\{t \geq 0 \colon X(t) \in \{a, b\}\}$. Let $\alpha := 2\,|\mu|\,/\sigma^2$. Show that

$$\mathbf{E}T = \frac{1}{\mu} \cdot \frac{b(1 - e^{\alpha a}) + a(e^{\alpha b} - 1)}{e^{\alpha b} - e^{\alpha a}}$$

Exercise 14 (Optional)**.** Let $\{X(t)\}_{t\geq 0} = \{\sigma B(t) + \mu t\}_{t\geq 0}$ be a standard Brownian motion with variance $\sigma^2 > 0$ and negative drift $\mu < 0$. Let $a < 0$. Let $T_a := \inf\{t \geq 0 \colon X(t) = a\}$. Let $\alpha := 2\,|\mu|\,/\sigma^2$. Show that

$$\mathbf{E}T_a = \frac{a}{\mu}.$$

Exercise 15 (Optional)**.** Write a computer program to simulate standard Brownian motion. More specifically, the program should simulate a random walk on $\mathbf{Z}$ with some small step size such as .002. (That is, simulate $B_k(t)$ when $k = 500^2$ and, say, $0 \leq t \leq 1$.)

Exercise 16 (Optional)**.** The following exercise assumes familiarity with Matlab and is derived from Cleve Moler's book, Numerical Computing with Matlab.

The file `brownian.m` plots the evolution of a cloud of particles that starts at the origin and diffuses in a two-dimensional random walk, modeling the Brownian motion of gas molecules.

(a) Modify `brownian.m` to keep track of both the average and the maximum particle distance from the origin. Using loglog axes, plot both sets of distances as functions of n, the number of steps. You should observe that, on the log-log scale, both plots are nearly linear. Fit both sets of distances with functions of the form $cn^{1/2}$. Plot the observed distances and the fits, using linear axes.

(b) Modify `brownian.m` to model a random walk in three dimensions. Do the distances behave like $n^{1/2}$?

The program `brownian.m` appears below.

```
% BROWNIAN   Two-dimensional random walk.
%   What is the expansion rate of the cloud of particles?

shg
clf
set(gcf,'doublebuffer','on')
delta = .002;
x = zeros(100,2);
h = plot(x(:,1),x(:,2),'.');
axis([-1 1 -1 1])
axis square
stop = uicontrol('style','toggle','string','stop');
while get(stop,'value') == 0
   x = x + delta*randn(size(x));
   set(h,'xdata',x(:,1),'ydata',x(:,2))
   drawnow
end
set(stop,'string','close','value',0,'callback','close(gcf)')
```