Please provide complete and well-written solutions to the following exercises.

Due February 21, in the discussion section.

Homework 6

Exercise 1. Let $\Omega = [0,1]$. Let $\mathbf{P}$ be the uniform probability law on Ω. Let $X\colon [0,1] \to \mathbf{R}$ be a random variable such that $X(t) = t^2$ for all $t \in [0,1]$. Let

$$\mathcal{A} = \{[0,1/4), [1/4,1/2), [1/2,3/4), [3/4,1]\}$$

Compute explicitly the function $\mathbf{E}(X|\mathcal{A})$. (It should be constant on each of the partition elements.) Draw the function $\mathbf{E}(X|\mathcal{A})$ and compare it to a drawing of X itself.

Now, for every integer $k > 1$, let $s = 2^{-k}$, and let $\mathcal{A}_k := \{[0,s), [s,2s), [2s,3s), \ldots, [1-2s, 1-s), [1-s,1)\}$. Try to draw $\mathbf{E}(X|\mathcal{A}_k)$. Convince yourself of the following fact (you can prove it if you want, but you do not have to): for every $t \in [0,1]$

$$\lim_{k\to\infty} \mathbf{E}(X|\mathcal{A}_k)(t) = X(t).$$

The purpose of this exercise is to demonstrate that $\mathbf{E}(X|\mathcal{A})$ is given by averaging X over each partition element, such that $\mathbf{E}(X|\mathcal{A})$ is constant on each partition element of $\mathcal{A}$.

Exercise 2. Let X be a random variable with finite variance, and let $t \in \mathbf{R}$. Consider the function $f\colon \mathbf{R} \to \mathbf{R}$ defined by $f(t) = \mathbf{E}(X-t)^2$. Show that the function f is uniquely minimized when $t = \mathbf{E}X$. That is, $f(\mathbf{E}X) < f(t)$ for all $t \in \mathbf{R}$ such that $t \neq \mathbf{E}X$. Put another way, setting t to be the mean of X minimizes the quantity $\mathbf{E}(X-t)^2$ uniquely.

The conditional expectation, being a piecewise version of taking an average, has a similar property. Let $A_1, \ldots, A_k \subseteq \Omega$ such that $A_i \cap A_j = \emptyset$ for all $i,j \in \{1,\ldots,k\}$ with $i \neq j$, and $\cup_{i=1}^k A_i = \Omega$. Write $\mathcal{A} = \{A_1, \ldots, A_k\}$. By definition, for each $1 \leq i \leq k$, $\mathbf{E}(X|\mathcal{A})$ is constant on A_i. Now, let Y be any other random variable such that, for each $1 \leq i \leq k$, Y is constant on A_i. Show that the quantity $\mathbf{E}(X-Y)^2$ is uniquely minimized by such a Y only when $Y = \mathbf{E}(X|\mathcal{A})$.

Exercise 3. Let $\Omega = [0,1]$. Let $\mathbf{P}$ be the uniform probability law on Ω. Let $X\colon [0,1] \to \mathbf{R}$ be a random variable such that $X(t) = t^2$ for all $t \in [0,1]$. For every integer $k > 1$, let $s = 2^{-k}$, let $\mathcal{A}_k := \{[0,s), [s,2s), [2s,3s), \ldots, [1-2s,1-s), [1-s,1)\}$, and let $M_k := \mathbf{E}(X|\mathcal{A}_k)$. Show that the increments $M_2 - M_1, M_3 - M_2, \ldots$ are orthogonal in the following sense. For any $i,j \geq 1$ with $i \neq j$,

$$\mathbf{E}(M_{i+1} - M_i)(M_{j+1} - M_j) = 0.$$

This property is sometimes called **orthogonality of martingale increments**. This property holds for many martingales, but we will not prove this.

Exercise 4. Let $X_0 = 0$. Let $(X_0, X_1, \ldots)$ such that $\mathbf{P}(X_i = 1) = \mathbf{P}(X_i = -1) = 1/2$ for all $i \geq 1$. For any $n \geq 0$, let $Y_n = X_0 + \cdots + X_n$. So, $(Y_0, Y_1, \ldots)$ is a symmetric simple random walk on $\mathbf{Z}$. Show that $Y_n^2 - n$ is a martingale (with respect to $(X_0, X_1, \ldots)$).

Exercise 5. Let $1/2 < p < 1$. Let $(X_0, X_1, \ldots)$ such that $\mathbf{P}(X_i = 1) = p$ and $\mathbf{P}(X_i = -1) = 1 - p$ for all $i \geq 1$. For any $n \geq 0$, let $Y_n = X_0 + \cdots + X_n$. Let $T_0 = \min\{n \geq 1 \colon Y_n = 0\}$. Prove that $\mathbf{P}_1(T_0 = \infty) > 0$. Then, deduce that $\mathbf{P}_0(T_0 = \infty) > 0$. That is, there is a positive probability that the biased random walk never returns to 0, even though it started at 0.

Exercise 6. Let $X_1, \ldots$ be independent identically distributed random variables with $\mathbf{P}(X_i = 1) = \mathbf{P}(X_i = -1) = 1/2$ for every $i \geq 1$. For any $n \geq 1$, let $M_n := X_1 + \cdots + X_n$. Let $M_0 = 0$. For any $n \geq 1$, define

$$W_n := M_0 + \sum_{m=1}^{n} H_m(M_m - M_{m-1}).$$

Show that if you have an infinite amount of money, then you *can* make money by using the double-your-bet strategy in the game of coinflips (where if you bet \$$d$, then you win \$$d$ with probability 1/2, and you lose \$$d$ with probability 1/2). For example, show that if you start by betting \$1, and if you keep doubling your bet until you win (which should define some betting strategy $H_1, H_2, \ldots$ and a stopping time T), then $\mathbf{E}W_T = 1$, for a suitable stopping time T.