Please provide complete and well-written solutions to the following exercises.

Due January 31, in the discussion section.

Homework 3

Exercise 1. Let x, y be points in the state space of a finite Markov Chain $(X_0, X_1, \ldots)$ with transition matrix P. Let $T_y = \min\{n \geq 1 \colon X_n = y\}$ be the first arrival time of y. Let j be a positive integer. Show that

$$P^j(x, y) \leq \mathbf{P}_x(T_y \leq j).$$

(Hint: can you induct on j?)

Exercise 2. Let x, y be any states in a finite irreducible Markov chain. Show that $\mathbf{E}_x T_y < \infty$. In particular, $\mathbf{P}_y(T_y < \infty) = 1$, so all states are recurrent.

Exercise 3 (Simplified Monopoly)**.** Let $\Omega = \{1, 2, \ldots, 10\}$. We consider Ω to be the ten spaces of a circular game board. You move from one space to the next by rolling a fair six-sided die. So, for example $P(1, k) = 1/6$ for every $2 \leq k \leq 7$. More generally, for every $j \in \Omega$ with $j \neq 5$, $P(j, k) = 1/6$ if $k = (j + i) \bmod 10$ for some $1 \leq i \leq 6$. Finally, the space 5 forces you to return to 1, so that $P(5, 1) = 1$. (Note that $\bmod 10$ denotes arithmetic modulo 10, so e.g. $7 + 5 = 2 \bmod 10$.)

Using a computer, find the unique stationary distribution of this Markov chain. Which point has the highest stationary probability? The lowest?

Compare this stationary distribution to the stationary distribution that arises from the doubly stochastic matrix: for all $j \in \Omega$, $P(j, k) = 1/6$ if $k = (j + i) \bmod 10$ for some $1 \leq i \leq 6$. (See Exercise 6.)

Exercise 4. Give an example of a Markov chain where there are at least two different stationary distributions.

Exercise 5. Is there a finite Markov chain where no stationary distribution exists? Either find one, or prove that no such finite Markov chain exists.

(If you want to show that no such finite Markov chain exists, you are allowed to just prove the weaker assertion that: for every stochastic matrix P, there always exists a nonzero vector π with $\pi = \pi P$.)

Exercise 6. Let P be the transition matrix for a finite Markov chain with state space Ω. We say that the matrix P is **doubly stochastic** if the columns of P each sum to 1. (Since P is a transition matrix, each of its rows already sum to 1.) Let π such that $\pi(x) = 1/|\Omega|$ for all $x \in \Omega$. That is, π is uniform on Ω. Show that $\pi = \pi P$.